OM

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ECE

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PM 1 (B)

Maths (Calculus)

$1 + x^2 + \frac{x^5}{2!}$

$+ \frac{x^6}{3!}$

$+ x^3 + \frac{x^{?}}{}$

CALCULUS:

→ Mean Value Theorem.
→ Definite Integrals.
→ Improper Integrals.
→ Partial Differentiation.
→ Multiple Integrals.
→ Vector Differentiation.
→ Vector Integration.
→ Fourier Series.

* Function:

→ A $f: A \to B$ if $\forall x \in A$ $\exists$ a unique $y \in B$ such that $f(x) = y$.

① Explicity function:

→ $z = f(x_1, x_2, x_3, \dots x_n)$

eg: $y = x(x-2)$.

$\Rightarrow y = f(x)$.

② Implicity function:

$\phi(z, x_1, x_2, \dots x_n) = c$.

E.g. $x^2 + xy + y^2 = c$.

$\Rightarrow \phi(x, y) = c$.

③ Composite function:

→ If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$.

* Some Special function:

① Even function:

→ $f(-x) = f(x)$.

E.g.: $\cos x$, $|x|$, ...

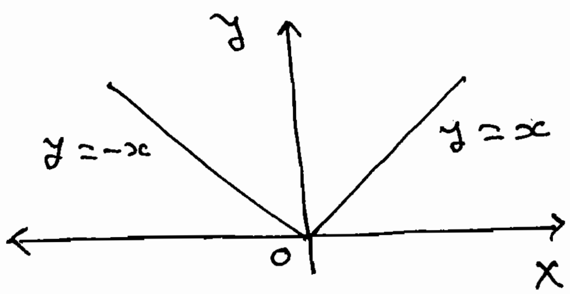
② Odd function:

→ $f(-x) = -f(x)$.

E.g.: $\sin x$, x, ...

③ Modulus function:

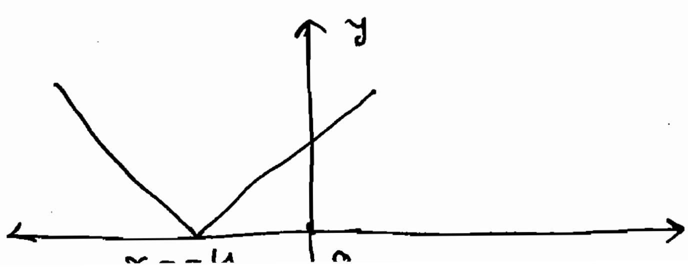
$$\rightarrow f(x) = |x| = \begin{cases} +x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$



* $\frac{d}{dx}|x| = \frac{|x|}{x}$ if $x \neq 0$.

→ $|x|$ is diffⁿ every where except $x = 0$. and contⁿ every where.

NOTE: $|x+a|$ is Not diffⁿ at $x = -a$.



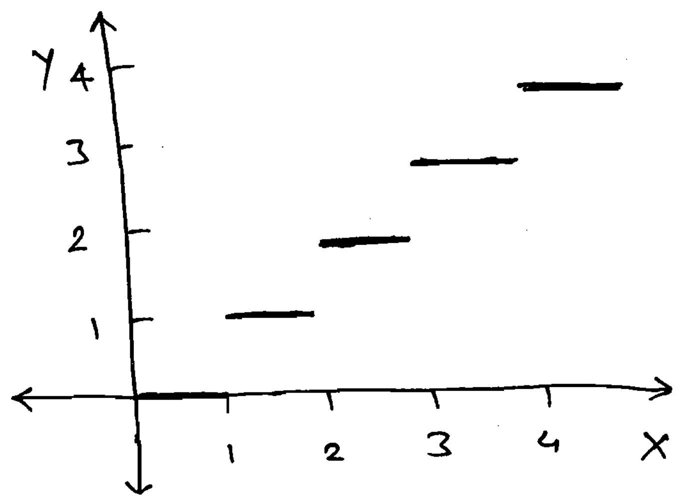
(OR) Step (OR) Bracket function:

$\rightarrow$ $f(x) = [x] = n \in \mathbb{Z}$.

where $n \le x < n+1$.

E.g. $[2.2] = 2$. | $[2] = 2$

$[2.999] = 2$ | $[-1.2] = -2$.



* Continuity of a function:

(i) At a point:

$\rightarrow$ $f(x) \rightarrow$ Cont $\rightarrow x = a$,

if $\boxed{\lim_{x \to a} f(x) = f(a).}$

(ii) In a Interval $[a, b]$:

$\rightarrow$ $f(x) \rightarrow$ Cont. $\rightarrow [a, b]$ if

(a)	$f(x)$ is Cont $\forall x \in (a, b)$.
(b)	$\lim_{x \to a^+} f(x) = f(a)$.
(c)	$\lim f(x) = f(b)$.

* Differentiation:

→ $f(x) \to$ diff $\to x = c$.

if $\boxed{\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c).}$

exist and finite.

$$\boxed{LHD = \lim_{h \to 0} \left[\frac{f(-h) - f(c)}{-h} \right].}$$

$$\boxed{RHD = \lim_{h \to 0} \left[\frac{f(-h) - f(c)}{h} \right].}$$

Eg. $f(x) = \sqrt[3]{x} \Rightarrow f'(x) = \frac{1}{3} x^{-2/3}$.

$\Rightarrow f'(0) = \infty$.

$f(x)$ is not diffⁿ at $x = 0$.

* ~~Mean Value Theorems.~~

Notes:

→ The necessary condition for a function to be diffⁿ at a point is the existance of finite LHD, finite RHD & equality of both of them.

→ every diffⁿ fⁿ is continous but a contⁿ fⁿ may not be ~~continous.~~ diffⁿ. fⁿ.

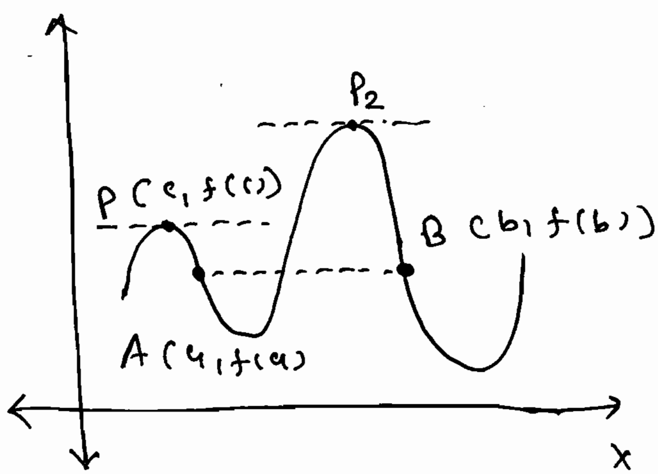
Mean Value Theorem:

* Rolle's Theorem:

→ Let $f(x)$ be defined in $[a,b]$ such that

① $f(x)$ is continous in $[a,b]$.
② $f(x)$ is diff^n in (a,b).
③ $f(a) = f(b)$.

→ Then ∃ (there exist) atleast one point $c \in (a,b)$ such that $\boxed{f'(c) = 0}$.



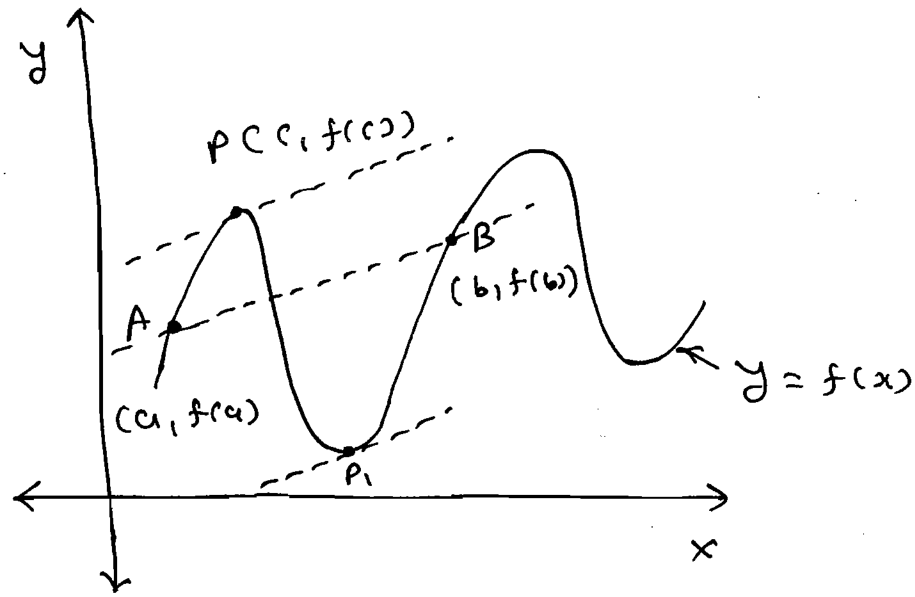
* Langrauges Mean Value Theorem:

→ Let $f(x)$ be defined in $[a,b]$ such that

(a) $f(x)$ is continous in $[a,b]$.
(b) $f(x)$ is diff^n in (a,b).

Then ∃ atleast one point $c \in (a,b)$ such

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}.$$



y
P(c, f(c))
B
(b, f(b))
A
(a, f(a))
y = f(x)
P1
x

Ex-1 The Mean value c for the fn
$f(x) = e^{x} [\sin x - \cos x]$. in the $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π. ✓

Ans: $f(x) = e^{x} [\sin x - \cos x]$

$\rightarrow$ $f'(x) = e^{x} [\sin x - \cos x] + e^{x} [\cos x + \sin x]$.

$f'(x) = 2e^{x} \cdot \sin x$.

$\therefore$ $f(x)$ is contn in $[a,b]$ &

$f(x)$ is diffn in (a,b).

Now, $f(a) = f(\frac{\pi}{4}) = 0$

$f(b) = f(\frac{5\pi}{4}) = 0$

$\therefore$ $f(a) = f(b)$.

So, by Rolle's theorem,

$\therefore$ $f'(c) = 0$.

$\therefore$ $f'(c) = 2e^{c} \cdot \sin c = 0$.

$\therefore$ $\sin c = 0$.

$\therefore$ $c = 0, \pm\pi, \pm 2\pi \ldots$

But $[\frac{\pi}{4}, \frac{5\pi}{4})$.

$\therefore$ $\boxed{c = \pi}$

Ex - 2 The Mean value c for the f^n

$f(x) = (x-1)^{2/3}$, $[1, 2]$ is — ?

(a) $27/8$, (b) $35/27$

(c) $35/29$, (d) not applicable.

Ans: $f(x) = (x-1)^{2/3}$.

① Continuity. ② Differentiation.

$\rightarrow$ $f'(x) = \frac{2}{3} \times (x-1)^{-1/3}$

$f'(x)$ is finite every where except $x = 1$.

$\therefore$ $f(x)$ is $diff^n$ in $(a, b) = (1, 2)$.

at $x = 1$.

$(\because x = 1$ is not in $(1, 2))$.

$\rightarrow$ for $Cont^n$;

$[1, 2]$ →
- $(1, 2)$ ✓
- LC at $x = 2$. ✓
- RC at $x = 1$. ✓

$f(1) = 0$, $RL = (1-1)^{2/3} = 0$.

So, $f(x)$ is $Cont^n$ at $x = 1$.

Now, $f(1) = 0$, $f(2) = 1$.

So, By Langrange's theorem. $c \in (1, 2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2-1} = \frac{2}{3[c-1]^{1/3}}$$

$$\therefore \frac{1-0}{1} = \frac{2}{3(c-1)^{1/3}}$$

$\therefore$ $c - 1 = 8/27$

$\therefore$ $\boxed{c = 35/27}$

Ex-3 The value ... $f(b) - f(a) = (b-a) f'(\xi)$ for the

☆ f^n $f(x) = Ax^2 + Bx + C$. in $[a, b]$.

(a) $\frac{b+a}{2}$ (b) $\frac{b-a}{2}$ (c) $\frac{b+a}{4}$ (d) $\frac{b-a}{4}$.

Ans: $f'(x) = 2Ax + B$.

$\therefore f'(x) = \frac{f(b) - f(a)}{b-a}$.

$\therefore 2A\xi + B = \frac{[b^2A + bB + C] - [Aa^2 + aB + C]}{b-a}$.

$\therefore 2A\xi + B = \frac{A(b^2 - a^2) + B(b-a)}{(b-a)}$

$\therefore 2A\xi + \cancel{B} = \frac{(b+a)A + \cancel{B}}{1}$.

$\therefore \boxed{\xi = \frac{b+a}{2}.}$

☆☆☆

Ex-4 The mean value C for the f^n $f(x) = 3/2\, x^2 - 5x + 8/3$ in the $[\frac{11}{2}, \frac{17}{2}]$. is ?

Ans (a) 5.75 (b) 6.5

(c) 7 (d) 7.75.

Ans: $f(x)$ is $cont^n$ and $diff^n$ in $[a, b]$ & (a, b) respectively.

$\therefore f(\frac{11}{2}) = 3/2 \times \frac{121}{4} - 5(\frac{11}{2}) + 8/3$.

$f(\frac{11}{2}) = \frac{363}{8} - \frac{55}{2} + 8/3$.

$f(\frac{11}{2}) = \frac{1089 - 660 + 64}{\ldots}$

∴ $\boxed{\text{mean value} = \frac{[illegible]}{2}}$. for 2^{nd} degree polynomial.

$$= \frac{11/2 + 17/2}{2}$$

$$= \frac{28}{4}$$

$$= 7.$$

NOTE: This applicable for 2^{nd} degree polynomial.

Ex-5 If $f: [-5, 5] \to R$ is a diff^n f^n. and $f'(x)$ doesn't vanish anywhere in $(-5, 5)$. Then.

$f'(c) = \frac{f(b) - f(a)}{b-a}$.

(a) $f(x)$ is not Cont^n in $[-5, +5]$.

✓ (b) $f(-5) \neq f(+5)$.

(c) $f(-5) = f(+5)$.

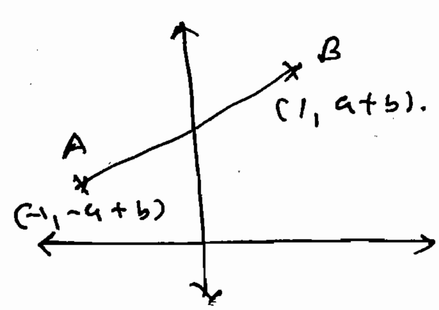
(d) a & b.

Ex-6 If $f(x) = ax + b$, $x \in [-1, 1]$ then a point $c \in (-1, +1)$. such that $2 \cdot f'(c) = f(1) - f(-1)$ is — ?

(a) $c = 0$ only. (b) $c = \pm \frac{1}{2}$ only.

(c) can be any c in $(-1, 1)$.

(d) doesn't exist.



Ans: $2 \cdot f'(c) = \frac{f(1) - f(-1)}{2}$.

∴ $2 \cdot (a) = \frac{a + b - (-a + b)}{2}$

∴ $\boxed{a = a}$ Ans: (c)

Satisfying LMVT in the given intervals.

(A) $f(x) = |x+2|$ in $[-2, 0]$. ✓

(B) $g(x) = 2 + (x-2)^{1/3}$ in $[1, 3]$. ×

(C) $h(x) = \log(x + x^3)$ in $[0, 2]$. ✓

(d) $p(x) = \begin{cases} 1+x^2, & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1. \end{cases}$ ×

(a) 0 (b) 1 (c) 2 (d) 3.

Ans:
(ii) $g'(x) = \frac{1}{3}(x-2)^{-2/3}$
×

$\Rightarrow g'(2) = \infty$

$\Rightarrow g(x)$ is not diff in $(1, 3)$.

(iii) $h'(x) = \frac{1}{1+x^2} \times 3x^2$.

(iv) $[0,1]$ → $(0,1)$.
×
→ RC at $x = 0$.
→ LC at $x = 1$.

$p(1) = 1$

$LL = 1 + (1)^2 = 2$.

$p(1) \neq LC$.

Ex-8 If $f'(x) = \frac{1}{5-x^2}$, $f(0) = 1$. then the Lower and upper boundary of $f(1) =$ __ ?

Ans: Let, $f(x)$ be defined in $[0,1]$.

By LMVT $\exists c \in (0,1)$ such that the

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}.$$

$$\Rightarrow f'(c) = f(1) - 1.$$

$\rightarrow \min\{f'(x)\} < f'(c) < \max\{f'(x)\}$.

$\therefore \frac{1}{5-0} < f(1)-1 < \frac{1}{5-1}$.

$\therefore \frac{1}{5} < f(1)-1 < \frac{1}{4}$.

$\therefore \boxed{\frac{6}{5} < f(1) < \frac{5}{4}}$.

☆ Cauchy's Mean Value Theorem:

→ Let $f(x)$ and $g(x)$ be defined in $[a,b]$ such that

(a) $f(x)$ and $g(x)$ is cont. in $[a,b]$.

(b) $f(x)$ & $g(x)$ are cont. in (a,b).

(c) $g'(x) \neq 0 \ \forall \ x \in (a,b)$.

Then $\exists$ atleast one point $c \in (a,b)$ such that $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$

$g(x) = x$
$g'(x) = 1$.
the LMVT.

Ex-1 The mean value c for the f^ns

$f(x) = 1/x$, $g(x) = \frac{1}{x^2}$. in $[1,2]$.

(a) $\frac{4}{3}$ ✓ (b) $5/3$ (c) $5/4$ (d) None.

Ans. $f'(x) = \frac{-1}{x^2}$, $g'(x) = \frac{-2}{x^3} \neq 0 \ \forall x \in (1,2)$.

is diff^n in (a,b).

By CMVT $\exists$ $c \in (1,2)$ such that

$$\therefore \frac{f'(c)}{g'(x)} = \frac{f(2) - f(1)}{g(2) - g(1)}.$$

$$\therefore \frac{-1/c^2}{-2/c^3} = \frac{\frac{1}{2}-1}{\frac{1}{4}-1}.$$

$$\therefore \boxed{c = 4/3.}$$

Ex-2 The Mean value c for the f^ns. $f(x) = e^x$, $g(x) = e^{-x}$ in $[0,1]$. is — ?

Ans: f(x) and g(x) is C & D.

$$\therefore \frac{f'(x)}{g'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)}.$$

$$\therefore \frac{e^c}{-e^{-c}} = \frac{e-1}{e^{-1}-1}.$$

$$\therefore -e^{2c} = \frac{e(e-1)}{(-e+1)}.$$

$$\therefore -e^{2c} = \frac{e^2-e}{1-e}.$$

$$\therefore e^{2c} = \frac{e\,\cancel{(e-1)}}{\cancel{(e-1)}}$$

$$\therefore e^{2c} = e.$$

$$\therefore 2c = 1.$$

$$\therefore \boxed{c = \frac{1}{2}}.$$

Taylor Series:

① $\boxed{f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \infty}$

is T.S.E. of $f(x)$ @ $x=a$.

② $\boxed{f(0) = f(0) + x\cdot f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \infty.}$

is T.S.E. of $f(x)$ about $x=0$.

also known as Maclarian Series.

NOTE:

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \infty.$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty.$

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty.$

④ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \infty.$

⑤ $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} - \cdots \infty\right].$

⑥ $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \infty.$

Taylor series expansion of $\log x$ about $x = 2$ is — ?

(a) $-\frac{1}{4}$ (b) $-\frac{1}{24}$

(c) $-\frac{1}{64}$ (d) None.

Ans: Coeff. of $(x-2)^4 = \frac{f^4(2)}{4!}$

$f(x) = \log x$

$f'(x) = \cancel{4(x-2)^3} = 1/x$

$f''(x) = \cancel{12(x-2)^2} = -1/x^2$

$f'''(x) = \cancel{24(x-2)} = +2/x^3$

$f^{iv}(x) = \cancel{24} -18/x^4$ $f^{iv}(2) = -6/x^4$

$\therefore$ Coeff. $= \cancel{\frac{24}{4 \times 3}} = \frac{-18}{16} \times \frac{1}{24} = \frac{-3}{192}$

Ex-2 The co-efficient of x^2 in the power series expansion of $e^{\cos 2x}$ in the assending power's of x is — ?

(a) $-1/2$ (b) $+1/2$.

(c) -2 (d) None.

Ans: Coeff. of $x^2 = \frac{f''(0)}{2!} = \frac{-4}{2} = -2$

$\therefore f(x) = e^{\cos 2x}$.

$\therefore f'(x) = e^{\cos 2x} \cdot (-2\sin 2x)$.

$f''(x) = -2[e^{\cos 2x} \cdot 2\cos 2x + \sin 2x \cdot [e^{\cos 2x} \cdot (-2\sin 2x)]$

$\therefore f''(0) = -2[2] = -4$. $\therefore$

Ex-3 The T. S. E. of $\tan x$ about ... ?

Ans:
$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x-\frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \dots \infty.$$

$\therefore$ $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1.$

$f'(x) = \sec^2 x. \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = 2$

$f''(x) = 2\sec x . \sec x . \tan x. \quad f''\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \cdot 1$

$= 2 \cdot 4.$

$\therefore$ $f''\left(\frac{\pi}{4}\right) = 4.$

$\therefore$ $f(x) = 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \dots$

Ex-4 Express $\cos^2 x$ as a power series using a Taylor series E. @ $x = 0$.

Ans: $f(x) = \cos^2 x.$

$\therefore$ $f'(x) = -\sin 2x. \quad \Rightarrow \quad f'(0) = 0$

$f''(x) = -2\cos 2x. \quad \Rightarrow \quad f''(0) = 2$

$f'''(x) = +4\sin 2x. \quad \Rightarrow \quad f'''(0) = 0.$

$f^{iv}(x) = +8\cos 2x \quad \Rightarrow \quad f^{iv}(0) = -8.$

$\therefore$ $f(x) = \cos^2 x = 1 + 0 - \frac{2x^2}{2!} + 0 + \frac{8x^4}{4!}.$

$f(x) = 1 - x^2 + \frac{x^4}{3} + \dots \infty.$

$\therefore$ $f(x) = \cos^2 x = \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} \dots \infty\right].$

$= \frac{1}{2}\left[1 + \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \dots \infty\right]\right]$

Ex - ⑤ The first four none zero terms in T.S.E. of $f(x) = e^x \cdot \cos x$ is — ?

(a) $1 + x - \frac{x^3}{3} + \frac{x^4}{24}$ (b) $1 + x + \frac{x^3}{3} + \frac{x^4}{4}$.

(c) $1 + x - \frac{x^3}{3} - \frac{x^4}{6}$ (d) $1 + x + \frac{x^3}{3} - \frac{x^4}{24}$.

Ans: $f_1(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \infty$

$f_2(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \infty$.

$\therefore f(x) = e^x \cdot \cos x = f_1(x) \cdot f_2(x)$.

$\cancel{= 1 + x - \frac{x^2}{2!} + \frac{x^2}{2!} - \frac{x^3}{2!} + \frac{x^3}{2!} + \frac{x^6}{6!}}$

$= 1 + x - \frac{x^2}{2!} + \frac{x^2}{2!} - \frac{x^3}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^4}{4} + \frac{x^4}{4!} + \cdots$

$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots \infty$.

Ex - 6 The ~~Leniar~~ Linear approximation to $x \cdot e^{-5x}$ around $x = 2$. is — ?

Ans: $f(x) = \underbrace{f(a) + (x-a) f'(a)}_{\text{Linear approx.}} + \frac{(x-a)^2}{2!} f''(a) + \cdots$

Square approx.

$f(x) = x \cdot e^{-5x}$

$\therefore f'(x) = e^{-5x} - 5x\, e^{-5x}$.

$\Rightarrow f'(2) = e^{-10} - 10 \cdot e^{-10} = -9e^{-10}$.

Linear approx. $= f(2) + (x-2) \cdot f'(2)$.

$= 2e^{-10} - (x-2) \cdot 9e^{-10}$.

☆ Definite Integrals.

⇒ **Theorem:-** Let $f(x)$ be Cont. in $[a,b]$ and $F(x)$ be the anti-derivative (integration) of $f(x)$ then.

$$\int_a^b f(x).dx = F(b) - F(a).$$

Note:

$$\boxed{\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)\,dx\right] = f(v)\frac{dv}{dx} - f(u).\frac{du}{dx}}$$

⇒ **Properties:**

① $$\int_a^b f(x).dx = -\int_b^a f(x).dx.$$

② If $c \in (a,b)$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x).dx + \int_c^b f(x)\,dx.$$

③ $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$ ✓

④ $$\int_a^b \frac{f(x)}{f(x)+f(b+a-x)}.dx = \frac{b-a}{2}.$$ ✓

⑤ $$\int_{-a}^{+a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) = \text{even}. \\ 0, & \text{if } f(x) = \text{odd}. \end{cases}$$ ✓

⑥ $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x).\,dx$, if $f(2a-x) = f(x)$.

$= 0$, if $f(2a-x) = -f(x)$.

⑦ $\int_0^a x.\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ if $f(a-x) = f(x)$.

⑧ $\int_0^{\pi/2} \sin^n x.\,dx = \int_0^{\pi/2} \cos^n x.\,dx$

$= \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{2}{3} \text{ (or) } \frac{1}{2}\right] K.$

Where $K = 1$ if n is odd.

$= \frac{\pi}{2}$ if n is even.

⑨ $\int_0^{\frac{\pi}{2}} \sin^m x.\,\cos^n x.\,dx.$ $\boxed{m, n \in z^+}$

$= \frac{[(m-1)(m-3)(m-5) \times \cdots 2 \text{ (or) } 1]\,[(n-1)(n-3)(n-5) \times \cdots \times 2 \text{ (or) } 1]}{(m+n)(m+n-2)(m+n-4) \times \cdots 2 \text{ (or) } 1} \times K.$

Where $K = \frac{\pi}{2}$, when m & n both are even.

$= 1$, otherwise.

Ex-1 $\lim_{x \to 0} \left[\dfrac{\int_0^{x^2} \cos t^2 \, dt}{x \cdot \sin x} \right] = \text{\_\_\_\_} ?$

(a) 0
(b) 1
(c) 3
(d) 2.

Ans: $\lim_{x \to 0} \left[\dfrac{\int_0^{0} \cos t^2 . dt}{0 \cdot \sin 0} \right] = \dfrac{0}{0}$

$\therefore$ L'Hospital rule,

$\therefore \lim_{x \to 0} \dfrac{\dfrac{d}{dx}\left[\int_0^{x^2} . \cos t^2 . dt \right]}{\dfrac{d}{dx} [x \cdot \sin x]}$.

$= \lim_{x \to 0} \left[\dfrac{\cos x^4 . (2x) . - (1)(0)}{\sin x + x \cos x} \right]$.

$= \lim_{x \to 0} \dfrac{2 \cos^4 x + \cancel{8x \sin} \; 2x [-\sin x^4] 4x^3}{\cos x + \cos x - x \sin x}$.

$= \dfrac{2(1) + 0}{1 + 1 - 0} = 1.$

Ex-2 $f(x) \to [1, 2]$.
$f'(x) \to [1, 2]$.

then $\int_1^2 f'(x) \, dx = \text{\_\_\_\_}$.

(a) $f(2)$ (b) $f(1)$ (c) 1 (d) 0.

Ans: $\int_1^2 f'(x) = f(2) - f(1)$.

but $f(2) = f(1)$.

$\therefore$ ans $= 0$.

Ex-1 $\displaystyle\int_0 \frac{}{1+\sqrt{\cot x}}\, dx$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \text{— (1)}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \text{— (2).}$$

$$\therefore \; 2I = \int_0^{\pi/2} 1.\ dx.$$

$\cancel{\boxed{I = 0}}$.

$$2I = \pi/2.$$

$$\therefore \boxed{I = \pi/4.}$$

Ex-2 $\displaystyle\int_0^{\pi} |\cos x|\, dx.$

Ans: $\displaystyle I = \int_0^{\pi} |\cos x|\, dx.$

$$= \int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x.\, dx.$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= 1 - 0 - [0 - 1]$$

$$\therefore \boxed{I = 2}$$

Ex-3 $\int_0 |x^2-3x+2|\,dx.$

$I = \int_0^2 |x^2-3x+2|\,dx$

$(x-2)(x-1)=0$

$= \int_0^1 x^2-3x+2 \, dx \overset{-}{+} \int_1^2 x^2-3x+2 + \int$

$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 - \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$

$= \cancel{\frac{1}{3}} - \cancel{\frac{3}{2}} + \cancel{2} - \frac{8}{3} + \frac{3}{2}\times\overset{2}{\cancel{4}} \overset{-}{\cancel{+}} 4 + \cancel{\frac{1}{3}} + \cancel{\frac{3}{2}} - \cancel{2}$

$\cancel{=} \cancel{-\frac{8}{3}} \cancel{+8-4}.$ $= \frac{2}{3} - 3 + \cancel{4} - \frac{8}{3} + 6 - \cancel{4}$

$\cancel{=} \cancel{-\frac{8}{3}} \cancel{+2}$ $= -\frac{6}{3} + 3$

$\cancel{=} -2/3.$ $= -2+3$

$= 1.$

Ex-4 $\int_0^n [x]\,dx = $ ____ .

(a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+1)}{2}$

(c) n (d) $n-1$.

Ans. $[x] = n, \quad x \le n.$

$\cancel{I =} \int_0^n n\,dx.$ $I = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx.$

$+ \dots + \int_{n-1}^{n} (n-1)\,dx.$

$\neq$

$I = 0 + 1 + 2 + 3 + \dots$

$= \frac{(n-1)(n-1+1)}{2}$

$\therefore \boxed{I = \frac{n(n-1)}{2}}$

Ex-5 $\int_0^1 x(1-x)^{99}\, dx$

$I = \int_0^1 (1-x)(1-\cancel{x}1+x)^{99}\, dx.$

$I = \int_0^1 (1-x)\, x^{99}.\, dx.$

$\therefore$

$\therefore$

$I = \int_0^1 (x^{99} - x^{100})\, dx.$

$\therefore I = \left[\frac{x^{100}}{100} - \frac{x^{101}}{101}\right]_0^1$

$= \frac{1}{100} - \frac{1}{101}.$

$\boxed{I = \frac{1}{10100}}$

Ex-6 $\int_0^{\pi/2} \log(\tan x)\, dx.$

$\therefore I = \int_0^{\pi/2} \log(\cot x)\, dx.$

$I = \int_0^{\pi/2} -\log(\tan x)\, dx$

$\therefore I = -I$

$\therefore 2I = 0$

$\underline{\underline{Ex-7}}$ $\int_0 \frac{\log(1+x)}{(1+x^2)} dx.$

$$I = \int_0^1 \frac{\log(1+x)}{(1+x^2)} . dx.$$

let, $x = \tan\theta$.

$\therefore dx = \sec^2\theta . d\theta$.

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} . \sec^2\theta . d\theta.$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta).$$

$$\therefore I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right).d\theta.$$

$$I = \int_0^{\pi/4} \log\left(1+\left[\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}.\tan\theta}\right]\right).$$

$$\therefore I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta.$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta.$$

$$\therefore I = \int_0^{\pi/4} \log 2 - \int_0^{\pi/2} \log(1+\tan\theta.d\theta.$$

$$\therefore 2I = \frac{\pi}{4}\log 2.$$

$$\boxed{I = \frac{\pi}{8}\log 2}$$

Ex-8 $I = \int_{-1} \frac{|x|}{x}.dx.$

Ans: $I = 0$ ($\because$ $|x|$ is even &
x is odd fn).

$$f(x) = \frac{|x|}{x}.$$

$$f(-x) = \frac{|-x|}{-x} = -\frac{|x|}{x} = -f(x).$$

Ex-9 $\int_{-1/2}^{+1/2} \cos x . \log\left(\frac{1+x}{1-x}\right).dx.$

Ans: $f(-x) = \cos(-x). \log\left(\frac{1-x}{1+x}\right).$

$$= \cos x \log\left(\frac{1+x}{1-x}\right)^{-1}.$$

$$f(-x) = -\cos x. \log\left(\frac{1+x}{1-x}\right)$$

$\therefore$ $f(-x) = -f(x).$ $\Rightarrow$ odd fn

$\therefore$ $\boxed{I = 0}$

Ex-10 $\int_0^{\pi} \frac{x \sec x.}{\sec x + \tan x}.dx$

$$I = \int_0^{\pi} \frac{\frac{x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}.dx.$$

$$= \int_0^{\pi} \frac{x}{1+\sin x}.dx.$$

$$\therefore I = \int^{\pi} \frac{\pi - x}{\quad}.dx$$

$$I = \pi \int_0 \frac{}{1+\sin x} \qquad \int_0 \frac{}{1+\sin x}$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\sin x} \cdot dx.$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} \cdot dx.$$

$$= \frac{\pi}{2} \left[\int_0^{\pi} \sec^2 x + \int_0^{\pi} \frac{(-\sin x)}{\cos^2 x} \cdot dx. \right.$$

$$= \frac{\pi}{2} \left\{ [\tan \pi]_0^{\pi} - \cancel{\cos} [\sec x]_0^{\pi} \right\}$$

$$= \frac{\pi}{2} \left[0 - [(-1) - 1] \right].$$

$$= \frac{\pi}{2} [2].$$

$$\therefore \boxed{I = \pi}$$

Ex - 11 $\displaystyle\int_0^{\pi/2} \sin^8 x . dx.$

Ans:

$$I = \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \left(\tfrac{1}{2}\right) \text{or} \left(\tfrac{2}{3}\right). \right] k$$

$$I = \left[\frac{8-1}{8} \times \frac{5}{6} \times \frac{3}{4} \right] \frac{\pi}{2} \qquad (8 \text{ is even}).$$

$$= \frac{7}{8} \times \frac{5}{8} \cdot \times \frac{\pi}{2}$$

$$= \frac{35}{64}.$$

$$\therefore \boxed{I = \frac{35\pi}{128}}.$$

Ex-12 $I = \int_0^{\pi/2} \cos^{11} x \, dx.$

$$\Rightarrow \quad I = \left[\frac{10}{11} \times \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}\right] \times 1.$$

Ex-13 $\int_0^{\pi/2} \sin^7 x \cdot \cos^9 x \cdot dx.$

$$I = \left\{ \frac{\overset{(7+1)}{[(m-1)(m-3)(m-5)\ldots\ldots(2) \text{ or } 1]} \; [(n-1)(n-3)(n-5)\ldots(2 \text{ or } 1)]}{(m+n)(m+n-2)(m+n-4)(m+n-6)\ldots\ldots} \right\} k$$

$$I = \left[\frac{(6 \times 4 \times 2)(8 \times 6 \times 4 \times 2 \times 1)}{(16) \times 14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2 \times 1}\right] \cdot 1.$$

Ex-14 $\int_0^{\pi/2} \sin^8 x \cdot \cos^3 x \cdot dx.$

Ans: $I = \frac{(7 \times 5 \times 3 \times 1)(2 \times 1)}{11 \times 9 \times 7 \times 5 \times 3 \times 2} \times 1. \; (k = 1).$

Ex-15 $\int_0^{\pi/2} \sin^6 x \cdot \cos^4 x \cdot dx$

Ans: $I = \frac{(5 \times 3 \times 1) \times (3 \times 1)}{10 \times 8 \times 6 \times 2} \times \frac{\pi}{2}$ $\quad k = 6$ & 4 is even.

$$\int_{-\pi} (\sin x)^6 dx.$$

$\longrightarrow \quad f(-x) = (\sin(-x))^6 = \sin^6 x.$

$$\therefore I = 2\int_0^{\pi} (\sin x)^6 . dx. \qquad \begin{aligned} &f(2a-x) \\ &= f(\pi-x) \\ &= \sin(\pi-x) \\ &= \sin x = f(x). \end{aligned}$$

$$= 4\int_0^{\pi/2} \sin^6 x . dx.$$

$$\therefore I = 4 \times \left[\frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \right] \times \frac{\pi}{2} \cancel{\times \tfrac{1}{2}}..$$

Ex-17 $\int_0^{\pi} \sin^4 x . \cos^3 x . dx.$

$$I = \int_0^{\pi} [\sin(\pi-x)]^4 . [\cos(\pi-x)]^3 . dx.$$

$$\therefore I = -\int_0^{\pi} \sin^4 x . \cos^3 x . dx.$$

$I = -I$

$\therefore 2I = 0$

$\boxed{I = 0}$

Ex-18 $\int_{-2\pi}^{2\pi} \sin^4 x . \cos^6 x . dx.$

$$I = 2\int_0^{2\pi} \sin^4 x . \cos^6 x . dx.$$

$$I = 2\int_0^{\pi} [\sin(\pi-x)]^4 . [\cos(\pi-x)]^6 . dx.$$

$$I = 4\int_0^{\pi} \sin^4 x \cdot \cos^6 x \cdot dx.$$

$$\therefore I = 8\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \cdot dx.$$

$$\therefore I = 8 \times \left[\frac{(3\times 1)(5\times 3\times 1)}{10\times 8\times 6\times 4\times 2}\right] \times \frac{\pi}{2}.$$

Ex-19 $\displaystyle\int_0^{\pi/6} \sin^4 6\theta \cdot \cos^3 3\theta \cdot d\theta$

Ans: ; $3\theta = t$

$\therefore 3\,d\theta = dt$

$$I = \frac{1}{3}\int_0^{\pi/2} \sin^4 2t \cdot \cos^3 t \cdot dt.$$

$$= \frac{1}{3}\int_0^{\pi/2} (2)^4 \cdot \sin^4 t \cdot \cos^7 t \cdot dt.$$

$$= \frac{16}{3} \times \left[\frac{(3\times 1)(6\times 4\times 2)}{11\times 9\times 7\times 5\times 3\times 1}\right] \times 1.$$

Improper Integral.

(1) **First kind:**

$\int_a^b f(x).\,dx$ if $a = -\infty$ (or) $b = \infty$ (or) both.

(2) **Second kind:**

$\int_a^b f(x)\,dx$ if when a & b are finite but $f(x)$ is infinite in $x \in [a,b]$.

e.g. ① $\int_0^1 \log(1-x).\,dx.$

② $\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\,dx.$

③ $\int_{-1}^{1} \frac{1}{x}.\,dx.$

* Convergence:

→ ① If $\int_a^b f(x)\,dx$ = finite then it is a convergent improper integral.

→ ② If $\int_a^b f(x)\,dx$ = Infinite then it is a Divergent improper integral.

Ex-1 find the convergence

$$\int_1^\infty \frac{1}{x\sqrt{x^2-1}} \cdot dx.$$

Ans: $I = \int_1^\infty \frac{1}{x\sqrt{x^2-1}} \cdot dx$

$= \lim_{a\to\infty} \int_1^a \frac{1}{x\sqrt{x^2-1}} \cdot dx.$

$= \lim_{a\to\infty} [\sec^{-1}a - \sec^{-1}1].$

$= \lim_{a\to\infty} \sec^{-1}a - 0.$

$= \sec^{-1}\infty.$

$= \pi/2.$

So, convergent.

Ex-2 $\int_0^\infty x \cdot \sin x \cdot dx = \_\_\_\_.$

Ans: $I = \int_0^\infty x \cdot \sin x \cdot dx$

$= [x(-\cos x) - (1)(-\cos x) + (-\sin x)]_0^\infty.$

$= \infty.$

→ So, ~~convergent~~. divergent.

Ex - 3 $\int_{1}^{0} \log\left(\frac{1}{x}\right) dx = \_\_\_\_\_$.

Ans:

$$I = -\int_{1}^{\infty} \log(x)\, dx.$$

$$I = -\lim_{b\to\infty} \int_{1}^{b} 1\cdot\log(x)\, dx.$$

$$= -\lim_{b\to\infty} \left[x\log x - \left(\frac{1}{x}\right)(x)\right]_{1}^{b} + x.$$

$$= -\lim_{b\to\infty} \left[b\log b - 1\right].$$

$$= -\infty.$$

→ so, Divergent.

Ex - 4 $\int_{-\infty}^{0} e^{2x}\cdot \cos 3x\, dx = \_\_\_\_\_$.

Ans:

$$= \left\{\frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right].\right\}$$

$$= \left\{\frac{e^{2x}}{13}\left[2\cos 3x + 3\sin 3x\right]\right\}_{-\infty}^{0}$$

$$= \frac{1}{13}\left[2\right] = \frac{2}{13}.$$ So, Convergent.

Ex - 5 $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx.$

$$I = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}}\cdot dx$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx + \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\cdot dx.$$

$$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot dx$$

$$= 2\left[\sin^{-1} x\right]_0^1$$

$$= 2 \times \frac{\pi}{2}$$

$$= \pi$$

So, Convergent

Ex-6 $\int_{-1}^{1} \frac{1}{x^2} dx =$ ——.

Ans:

$$= \int_{-1}^{0} \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} \cdot dx.$$

$$= \left[\frac{-1}{x}\right]_{-1}^{0} + \left[\frac{-1}{x}\right]_0^1.$$

Both are divergent so,

$$= -\infty + \infty$$

$$= \infty.$$

So, divergent

Ex-7 $\int_0^3 \frac{1}{x^2-3x+2} dx =$ ——.

Ans:

$$I = \int_0^3 \frac{1}{(x-1)(x-2)} \cdot dx \cdot = \text{——}.$$

$$I = \int_0^1 + \int_1^2 + \int_2^3$$

$\int \;\; x-1 \;\; x-2 \qquad \int \;\; x-2 \;\; x-1$

$$= \log\left(\frac{x-2}{x-1}\right).$$

$$= \log\left(\frac{x-2}{x-1}\right)_0^1 + \log\left(\frac{x-2}{x-1}\right)_1^2 + \log\left(\frac{x-2}{x-1}\right)_2^3$$

$$= \infty.$$

So, Divergent.

* Comparision Test:

☆ Method I:

→ Let, $0 \le f(x) \le g(x)$ then

(i) $\int_a^b f(x)\,dx$ converges if $\int_a^b g(x)\,dx$ is convergent.

(ii) $\int_a^b g(x)\,dx$ diverges if $\int_a^b f(x)\,dx$ is divergent.

→ $x <$ finite ⇒ x is finite.
$x >$ Infinite ⇒ x is infinite.

→ $\left.\begin{array}{l} x > \text{finite} \\ x < \text{infinite} \end{array}\right\}$ x may be finite or infinite.

⇒ Method II [Limit form]

→ For first kind :

→ Let $f(x)$ and $g(x)$ be two +ve functions such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = l \left[\begin{array}{c} \text{Non-zero} \\ \text{finite} \end{array} \right].$$

then $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ both converge (or) diverge together.

→ For second kind:

ⓐ If $f(x) \to \infty$ as $x \to a$.

then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = l.$

(OR)

ⓑ If $f(x) \to \infty$ as $x \to b$

then $\lim_{x \to b^-} \frac{f(x)}{g(x)} = l.$

$\boxed{\text{NOTE:}}$

→ $\int_a^\infty \frac{1}{x^p}\,dx$ is

$$= \begin{cases} \text{convergent} & \text{if } p > 1. \\ \text{Divergent} & \text{if } p \leq 1. \end{cases}$$

Ex-1 $\int_1^{\infty} e^{-x^2} dx =$ ______.

Ans: $e^{x^2} \geq e^{x}$, $\forall x \geq 1$.

$\therefore e^{-x^2} \leq e^{-x}$.

$$I = \int_1^{\infty} e^{-x} \, dx = \left[\frac{e^{-x}}{-1}\right]_1^{\infty} = \frac{1}{e}$$

So, Convergent

Ex-2 $\int_2^{\infty} \frac{1}{\log x} dx = \underline{\text{Divergent}}$

Ans: $\log x < x$ $\forall x \geq 2$.

$\therefore \frac{1}{\log x} > \frac{1}{x}$.

$$\int_2^{\infty} \frac{1}{x} dx = \log x\Big]_2^{\infty} = \infty.$$ So, divergent.

Ex-3 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x\sqrt{x}} . dx = \underline{\text{Convergent}}$.

Ans: $\frac{\sin x}{x} \leq 1$.

$\therefore \frac{\sin x}{x\sqrt{x}} \leq \frac{1}{\sqrt{x}}$.

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{x}} \cdot dx = 2\sqrt{x}\Big]_0^{\frac{\pi}{2}} = 2\sqrt{\frac{\pi}{2}}.$$

So, convergence.

Ex-4 $\int_1 \frac{1}{x^2[e^{-x}+1]} \cdot dx =$ convergent.

Ans: let, $g(x) = \frac{1}{x^2}$. $\Rightarrow g(x) = \frac{1}{x^2}$

$p=2>1$ so, convergent

$\therefore \frac{f(x)}{g(x)} = \frac{1}{e^{-x}+1}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{0+1} = 1.$$

so, convergent

Ex-5 $\int_1^\infty \frac{x \tan^{-1}x}{\sqrt{4+x^3}} dx =$ divergent

Ans: $f(x) = \frac{x \tan^{-1}x}{\cancel{x}\sqrt{x}\left[\sqrt{\frac{4}{x^3}+1}\right]}$.

$$= \frac{\tan^{-1}x}{\sqrt{x}\left[\sqrt{\frac{4}{x^3}+1}\right]}.$$

$g(x) = \frac{1}{\sqrt{x}} \rightarrow$ div.

$$\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}} = \frac{\pi/2}{\sqrt{0+1}} = \frac{\pi}{2}.$$

divergent

Ex - 6 $\int_1 \frac{\sqrt{x}}{\log x} \cdot dx = $ Diverge.

Ans: Let, $g(x) = \frac{1}{x \log x}$.

$$\frac{f(x)}{g(x)} = x\sqrt{x}.$$

$$\Rightarrow \lim_{x \to 1^+} \frac{f(x)}{g(x)} = 1.$$

$$\int_1^2 \frac{1}{x \log x} dx, \quad \text{Let } \log x = t.$$

$$\rightarrow \frac{1}{x} \cdot dx = dt.$$

$$= \int_0^{\log_e 2} \frac{1}{t} dt = \log t \Big]_0^{\log_e 2}$$

$$= \infty.$$

So, Divergence

Ex - 7 How many of the following integrals are divergent

(1) $\int_0^1 \frac{1}{\sqrt{x + 4x^3}} dx$

(2) $\int_1^\infty \frac{x^4}{[1+x^3]^{5/2}} dx$

(3) $\int_1^\infty \frac{1}{\sqrt{x}\,[1+x^{1/3}]} dx$.

(a) 0 (b) 1 (c) 2 (d) 3.

$\rightarrow$ (1) $\int_0^1 \frac{1}{\sqrt{x + 4x^3}} dx$.

$$\sqrt{x + 4x^3} = \sqrt{x} \cdot \sqrt{1 + 4x^2}.$$

$$\sqrt{1+4x^2} \geq 1$$

$$\Rightarrow \sqrt{\cdots} \geq \sqrt{x}.$$

$\therefore \int_0 \frac{1}{\sqrt{x}} \cdot dx = \ldots x]_0$

Convergent.

2) $\int_1^{\infty} \frac{x^4}{(1+x^3)^{5/2}} \cdot dx.$

$\rightarrow \frac{1}{x^p} = \frac{1}{x^{\frac{15}{2}-4}} = \frac{1}{x^{7/2>1}} \rightarrow$ conv.

3) $\int_1^{\infty} \frac{1}{\sqrt{x}(1+x^{1/3})} dx \rightarrow \frac{1}{x^p} = \frac{1}{x^{5/6-0}} = \frac{1}{x^{5/6<1}}$

$\rightarrow$ div.

Ex 8 $\int_0^1 x \log x . dx =$ _____

Ans: (a) 0 (b) $\frac{1}{4}$ (c) $-\frac{1}{4}$ (d) ∞.

$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \cdot dx.$

$= \frac{x^2}{2} \cdot \log x - \frac{x^2}{4} \Big]_0^1$

$= -\frac{1}{4} - \lim_{x \to 0} \left[\frac{x^2}{2} \cdot \log x \right].$

$= -\frac{1}{4} - \frac{1}{2} \lim_{x \to 0} \frac{\log x}{(1/x^2)} \cdot 0 \ (\frac{0}{0}).$

$= -\frac{1}{4} - \frac{1}{2} \lim_{x \to 0} \frac{(1/x)}{-2/x^3}.$

$= -\frac{1}{4} - 0$

$= -\frac{1}{4}.$

* Gamma function

$$\rightarrow \boxed{\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx. \qquad (n>0).}$$

NOTE: ① $\Gamma 1 = 1$ ② $\Gamma \frac{1}{2} = \sqrt{\pi}$.

③ $\Gamma n+1 = n \Gamma n \quad \forall n > 0$.

④ $\Gamma n+1 = n! \quad \forall n \in z^+$.

⑤ $\int_0^\infty e^{-ax} \cdot x^{n-1} \cdot dx = \frac{\Gamma n}{a^n}$.

Ex-1 $\int_0^\infty e^{-x^2} \cdot dx. = \frac{\pi/4}{}$.

Ans: $x^2 = t$ $\quad x \to 0 \Rightarrow t \to 0$

$2x \cdot dx = dt$ $\quad x \to \infty \Rightarrow t \to \infty$

$dx = \frac{dt}{2\sqrt{t}}$.

$$I = \int_0^\infty \frac{1}{2} e^{-t} \cdot dt \cdot t^{-1/2}.$$

$$I = \frac{1}{2} \int_0^\infty e^{-t} \cdot dt. \; t^{\frac{1}{2}-1}.$$

$n = 1/2$

$\therefore I = \frac{1}{2} \Gamma 1/2$

$= \frac{1}{2} \times \sqrt{\pi}$

$$\boxed{I = \frac{\sqrt{\pi}}{2}}$$

NOTE: $\int_{-\infty}^{\infty} e^{-x^2} \cdot dx$

$$= 2\int_0^\infty e^{-x^2} \cdot dx$$

$$= 2 \times \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi}.$$

② $\int_0^1 (x \log x)^4 \cdot dx =$ ______.

Ans: Let $\log_e x = -t.$

$$\therefore \quad x = e^{-t}$$

$$dx = -e^{-t} \cdot dt$$

$$x = 0 \rightarrow t = \infty.$$

$$x = 1 \rightarrow t = 0.$$

$$I = \int_\infty^0 (e^{-t} \cdot t)^4 \cdot (-e^{-t})\, dt.$$

$$= \int_0^\infty e^{-5t} \cdot t^4 \cdot dt.$$

$$= \int_0^\infty e^{-5t} \cdot t^{5-1} \cdot dt.$$

$$= \frac{\Gamma 5}{5^5}$$

$$= \frac{4!}{5^5} = \frac{24}{5^5}.$$

Ex-5 $\int_0 5 \, .dx =$ ____.

Ans: let, $5^{-4x^2} = e^{-t}$

$\therefore \quad -4x^2 \log_e 5 = -t.$

$\therefore$

$$x^2 = \frac{t}{4\log_e 5}.$$

$$\therefore \quad x = \frac{\sqrt{t}}{2\sqrt{\log_e 5}}.$$

$$\therefore \quad dx = \frac{1}{2\sqrt{t}\cdot\sqrt{\log_e 5}\times 2}.dt.$$

$$I = \int_0^\infty e^{-t}.\frac{1}{4\sqrt{\log_e 5}}.\ t^{-1/2}.\ dt$$

$$= \frac{1}{4\sqrt{\log_e 5}} \times \int_0^\infty e^{-t}.\ t^{\frac{1}{2}-1}.dt.$$

$$= \frac{\Gamma\frac{1}{2}}{4\sqrt{\log_e 5}}.$$

$$\boxed{I = \frac{\sqrt{\pi}}{4\sqrt{\log_e 5}}.}$$

* Beta function:

$\therefore$ $$\boxed{B(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1}\, dx \quad (m,n>0).}$$

$\therefore$ **NOTE:**

$\rightarrow$ $\beta(m,n) = \beta(n,m).$

$\rightarrow$ $$\boxed{\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}}$$

$\rightarrow$ $$\boxed{\begin{aligned} \beta(m,n) &= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \\ &= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \cdot dx. \end{aligned}}$$

$\rightarrow$ $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta.$$

i.e. :→ $$\boxed{\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right).}$$

✓

Ex - 2 $\int_0 x^7 (16-x^4) \, dx = \_\_\_\_\_$

Ans: $\int_0^2 x^7 \cdot (16-x^4)^{10} \, dx$

let, $x^4 = 16t$

$\therefore 4x^3 \cdot dx = 16 \, dt$

$x^3 \cdot dx = 4 \, dt$

$$I = 4\int_0^1 \cancel{x^4}\, 16t\,(16-16t)^{10} \cdot dt$$

$$= 4 \times 16^{11} \int_0^1 t^{2-1} \cdot (1-t)^{11-1} \cdot dt$$

$$= 4 \times 16^{11} \times \beta(2, 11)$$

$$= 4 \times 16^{11} \times \frac{\Gamma 2 \times \Gamma 11}{\Gamma 13}$$

$$= 4 \times 16^{11} \times \frac{1! \times 10!}{12!}$$

$$I = \frac{4 \times 16^{11}}{12 \times 11}.$$

Ex-3 $\int_0^\infty \frac{x^3(1+x^5)}{(1+x)^{13}} \, dx = \_\_\_\_\_$

Ans: $I = \int_0^\infty \frac{x^3}{(1+x)^{13}} \, dx + \int_0^\infty \frac{x^8}{(1+x)^{13}} \cdot dx$

$$= \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}} \, dx + \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}} \cdot dx$$

$= \beta(4,9) + \beta(9,4)$

$= 2\,\beta(4,9).$

$= 2 \times \frac{\Gamma 4 \cdot \Gamma 9}{\Gamma 13}.$

$\therefore I = 2 \times \left[\frac{3! \times 8!}{12!}\right].$

Ex-3 $\displaystyle\int_0^{\infty} \left[\frac{x}{1+x^2}\right]^3 dx.$

Ans: Let, $x = \tan\theta.$

$\therefore dx = \sec^2\theta \cdot d\theta.$

$$I = \int_0^{\frac{\pi}{2}} \left[\frac{\tan\theta}{\sec^2\theta}\right]^3 \cdot \sec^2\theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{\sec^4\theta} \cdot d\theta.$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\theta \cdot \cos\theta \cdot d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3+1}{2}, \frac{1+1}{2}\right).$$

$$= \frac{1}{2}\beta(2,1).$$

$$= \frac{1}{2}\left[\frac{1\times1}{2}\right]$$

$$= 1/4$$

→ If $z = f(x, y)$ then

$$\therefore \boxed{\frac{\partial z}{\partial x} = z_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}}$$

$$\therefore \boxed{\frac{\partial z}{\partial y} = z_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}}$$

* Homogenius function:

→ E.g. ① $2x + 3y$

② $x^3 z - 2x^2 y^2 + 4xyz^2$

③ $\frac{x^2 y - xy^2}{\sqrt{x} + \sqrt{y}}$, $n = 3 - \frac{1}{2} = \frac{5}{2}$.

④ $u = \cos\left(\frac{x^2 - 2y^2}{\sqrt{x} + y}\right)$ → Non-homo. ($\because$ degree n is not zero of inner f^n).

⑤ $z = \log(x/y)$.

⑥ e^x, $\sin x$, $\log(1+x)$ -- are not homogenius f^n.

NOTE:

→ If $\boxed{f(kx, ky) = k^n f(x, y).}$ then $f(x, y)$ is a H.F. with degree 'n'.

→ If $f(x, y)$ is H.F. with degree 'n' then

$$\therefore \boxed{f(x, y) = \begin{cases} x^n \phi(y/x) \\ y^n \psi(x/y) \end{cases}}$$

Euler's Theorem.

→ If $f(x,y)$ is a homogeneous function with n. then

ⓐ $$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

ⓑ: $$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.}$$

NOTE:

→ If $u(x,y) = f(x,y) + g(x,y)$ where f and g are H.F.s with degree m and n respectively. then

ⓐ $$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mf + ng.}$$

ⓑ $$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g.}$$

→ If $f(u)$ is a H.F. with degree 'n' of two variable x & y then

ⓐ $$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = F(U)}$$

ⓑ $$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = F(u)\left[F'(u)-1\right]}.$$

Differentiation :-

→ If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then the total derivate of 'z' w.r.t. 't' is

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

→ Total differentiation of $z = f(x, y)$. is

$$\boxed{dz = \frac{\partial f}{\partial x} . dx + \frac{\partial f}{\partial y} . dy}$$

NOTE:

→ If $f(x, y) = c$ is an impicity f^n then

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

→ If $z = f(x, y)$ where $x = \phi(u, v)$ & $y = \psi(u, v)$

$$\therefore \boxed{\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial v}}$$

Ex-1 If $W = x^2+y^2$, $x = \frac{t^2-1}{t}$, $y = \frac{t}{t^2+1}$ then $\frac{dw}{dt}$ at $t=1$.

Ans:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

$$= 2x \times \left(\frac{t(2t) - t^2+1}{t^2}\right) + 2y \times \frac{(t^2+1)(1) - (t)(2t)}{(t^2+1)^2}.$$

$$= 2x \times \left(\frac{t^2+1}{t^2}\right) + 2y \frac{(1-t^2)}{(t^2+1)^2}.$$

$$\frac{dw}{dt} = \left(\frac{t^2-1}{t}\right) \times \left(\frac{t^2+1}{t^2}\right) + 2\left(\frac{t}{t^2+1}\right)\left(\frac{1-t^2}{(t^2+1)^2}\right)$$

$$\therefore \left.\frac{dw}{dt}\right|_{t=1} = 0 + 0 = 0.$$

Ex-2 The total derivative of x^3y^2 w.r.t. 'y' where, x & y are connected by the relation $x^3+y^3-3xy=0$. is ______

Ans: Let, $u = x^3y^2$.

$$\therefore \frac{du}{dy} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial y} + \frac{\partial u}{\partial y} \times \frac{dy}{dy}.$$

$$\therefore \frac{du}{dy} = 3x^2y^2 \times \cancel{(3x^2y - 3y)} \ \partial x/\partial y + 2x^3y$$

$$\therefore \frac{du}{dy} = 3x^2y^2\left(-\frac{f_y}{f_x}\right) + 2x^3y.$$

$$= 3x^2y^2\left(\frac{-(3y^2-3x)}{3x^2-3y}\right) + 2x^3y.$$

Ex-3 If $u = f(2x-3y, \ldots$

then $6u_x + 4u_y =$ ______ .

Ans:

$$p = 2x - 3y \qquad u = f(p, q, r).$$

$$q = 3y - 4z$$

$$r = 4z - 2x$$

$$\therefore u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x}.$$

$$= 2u_p + 0 - 2u_r.$$

$$u_x = 2u_p - 2u_r$$

$$\therefore u_y = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}.$$

$$\therefore u_y = -3u_p + 3u_q$$

$$\therefore 6u_x + 4u_y = \cancel{12u_p} - 12u_r - \cancel{12u_p} + 12u_q.$$

$$= 12(u_q - u_r)$$

$$\therefore u_z = 0 + -4u_q + 4u_r.$$

$$= -4(u_q - u_r).$$

$$-3u_z = -12(u_q - u_r)$$

$$\boxed{-3u_z = \cancel{12}\ 6u_x + 4u_y.}$$

ans : (c) $-3u_z$.

Ex-9 If $V = r^{n}$, $r^2 = x^2 + y^2 + z^2$

$V_{xx} + V_{yy} + V_{zz} = \_\_\_\_$.

Ans: $V = r^n$.

$\therefore V_x = n \cdot r^{n-1} \cdot \frac{\partial r}{\partial x}$.

$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2}x$.

$\therefore \frac{\partial r}{\partial x} = x/r$.

$\therefore V_x = n \cdot r^{n-2} \cdot x$

$\cancel{V_{xx} = n(n-1) \cdot r^{n-2} \cdot \frac{\partial r}{\partial x} + n \cdot r^{n-3}}$

$\cancel{= n(n-1) \cdot r^{n-2}}$

$\therefore V_{xx} = n \cdot r^{n-2}(1) + n(n-2) \cdot r^{n-3} \cdot \frac{\partial r}{\partial x}$.

$\therefore V_{xx} = n \cdot r^{n-2} + n(n-2) \cdot r^{n-3} \cdot x/r \cdot x$

$V_{xx} = n \cdot r^{n-2} + n(n-2) \cdot r^{n-4} \cdot x^2$

$\therefore V_{yy} = n \cdot r^{n-2} + n(n-2) \cdot r^{n-4} \cdot y^2$

$\therefore V_{zz} = n \cdot r^{n-2} + n(n-2) \cdot r^{n-4} \cdot z^2$.

$\therefore V_{xx} + V_{yy} + V_{zz} = 3n \cdot r^{n-2} + n(n-2) \cdot r^{n-4} (r^2)$.

$= 3n \cdot r^{n-2} + n(n-2) \cdot r^{n-2}$.

$= r^{n-2} [3n + n^2 - 2n]$.

$= n(n+1) r^{n-2}$.

Ex-5 If

$x u_x + y u_y + z \cdot u_z = \_\_\_\_ .$

Ans: $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = n u.$

$n = 0$

so Ans $= 0$.

Ex-6 If $u = \frac{x^2 y}{x^{5/2} + y^{5/2}}$. then $x^2 . u_{xx} + y^2 u_{yy}$

$+ 2xy u_{xy} = \_\_\_\_ .$

Ans: $n = 3 - 5/2 = 1/2.$

$\therefore x^2 . u_{xx} + y^2 u_{yy} + 2xy u_{xy} = \left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right) . u.$

$= -\frac{1}{4} u .$

Ex-7 If $u = \tan^{-1}\left[\frac{x^3 - 3y^3}{x + 2y}\right]$ then $x u_x + y u_y = \_\_\_\_ ?$

Ans: $F(u) = \tan u.$

$f'(u) = \sec^2 u.$

$n = 3 - 1 = 2$

$\therefore x u_x + y u_y = n . \frac{f(u)}{f'(u)}.$

$= 2 . \tan u . \cos^2 u.$

$= 2 . \sin u . \cos u .$

$\neq \cancel{2 \sin u}.$

$= \sin 2u .$

$\therefore x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = F(u) \left[F'(u) - 1 \right]$

$= \sin(2u) \left[2 \cos 2u - 1 \right].$

Ex- 8 If $u = \log\left(\frac{x^2+y^2}{x-y}\right)$, $x^2 u_{xx} + 2xy\, u_{xy}$
$+ y^2 u_{yy} = ?$

Ans:
$$f(u) = e^u.$$
$$n = 2-1 = 1.$$
$$f'(u) = e^u.$$
$\therefore$

$$\therefore \quad x u_x + y u_y = n \frac{f(u)}{f'(u)} = 1.$$
$$F(u) = 1$$

$$\therefore \quad x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$$
$$= F(u)\,[F'(u) - 1]$$
$$= 1\,[0-1]$$
$$= -1.$$

Ex- 9 If $z = \sin^{-1}\left[\frac{x^{1/4} - y^{1/4}}{x^{1/6} + y^{1/6}}\right]$ then

$$x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = \text{—— ?}$$

Ans: $$n = 1/6 - 1/4 = \frac{4-6}{24} = +\frac{2}{24} = +\frac{1}{12}.$$

$$f(u) = \sin z$$
$$f'(z) = \cos z$$

$$\therefore \quad x z_x + y z_y = n \frac{f(u)}{f'(u)}$$

$$= +\frac{1}{12} \tan z.$$

$$F(u) = +\frac{1}{12} \tan z.$$

$$F'(u) = +\frac{1}{12} \sec^2 z.$$

Now, $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy}$

$$= F(u)\,[F'(u) - 1]$$

$$= +\frac{1}{12}\tan z\left[+\frac{1}{12}[\sec^2 z] - 1\right] - 1].$$

$$\cancel{= -\frac{1}{12}\tan z\left[-\frac{\tan^2 z}{12} - 1\right].}$$

$$\cancel{= \frac{\tan 3z}{144} + \frac{\tan z}{12}.}$$

$$= +\frac{1}{12}\tan z\left[\frac{\sec^2 z - 1 + 11}{12}\right].$$

$$= \frac{\tan z}{144}\left[\tan^2 z - 11\right].$$

Ex-10 $\cancel{z}\, z = f(y/x) + \sqrt{x^2+y^2}$.

$x^2 z_{xx} + 2xy.\, z_{xy} + y^2 z_{yy} =$ ——.

Ans: $z = f(y/x) + \sqrt{x^2+y^2}$

$f(x,y) = f(y/x) \Rightarrow m = 0$

$\therefore$ $g(x,y) = \sqrt{x^2+y^2} \Rightarrow n = \cancel{0}\ 1$

$$x^2 z_{xx} + 2xy.\, z_{xy} + y^2 z_{yy} = m(m-1)f + n(n-1)g$$

$$= 0 + 0 = 0.$$

$\boxed{\text{NOTE:}}$

→ Every stationary point is not an extreme point but every extreme point is a stationary point.

Ex-1 $f(x) = 3x^4 - 4x^3 + 10$ has a minimum value at $x =$ ——.

Ans: $f(x) = 3x^4 - 4x^3 + 10.$

$\therefore \quad f'(x) = 12x^3 - 12x^2$

$f'(x) = 0.$

$12x^2(x-1) = 0..$

$\therefore \quad x = 0$, (or) $x = 1.$

Now, $f''(x) = 36x^2 - 24x.$

$f''(0) = 0.$

$f''(1) = 36 - 24 = 12 > 0$

So, $\boxed{x = \cancel{0}1}$ ans

$\therefore \quad f'(x) = 12x^2(x-1).$

For $x < 0, \quad f'(x) < 0.$

$x > 0. \quad f'(x) \cancel{>} 0.$

no. extreme at $x = 0.$

Maxima & Minima.

* For f^n of one Variable:

① $\boxed{f(x) \longrightarrow \max \longrightarrow x = c \text{ if } \exists\ \delta > 0 \text{ such that } |x-c| < \delta \Rightarrow f(x) \le f(c).}$

② $\boxed{f(x) \longrightarrow \min \rightarrow x = c \text{ if } \exists\ \delta/k\ \delta > 0. \text{ such that } |x-c| < \delta \Rightarrow f(x) \ge f(c).}$

* Method:

→ find $f'(x)$.

→ Equate $f'(x)$ to zero for obtaining the Stationary point.

→ At each St. pt. find $f''(x)$.

ⓐ If $f''(x_0) > 0 \Rightarrow \min.$

ⓑ If $f''(x_0) < 0 \Rightarrow \max.$

ⓒ If $f''(x_0) = 0.$

(i) For $x < x_0$, $f'(x) > 0$. and for $x > x_0$, $f'(x) < 0$. $\}$ → max.

(ii) For $x < x_0$, $f'(x) < 0$. and for $x > x_0$, $f'(x) > 0$ $\}$ → min

(iii) For $x < x_0$ and $x > x_0$ if $f'(x) > 0$ (or) $x < x_0$ and $x > x_0$ if $f'(x) < 0$ $\}$ no extreme value.

Ex-2 $f(x) = \frac{e^{\sin x}}{e^{\cos x}}, \quad x \in R$

(a) $e^{\sqrt{2}}$ (b) $e^{-\sqrt{2}}$. (c) $e^{1/\sqrt{2}}$. (d) $e^{-1/\sqrt{2}}$.

Ans: $f(x) = e^{\sin x - \cos x}$.

$\therefore f'(x) = e^{\sin x - \cos x} \cdot [\cos x + \sin x] = 0.$

let, $g(x) = \sin x - \cos x$.

$g'(x) = \cos x + \sin x. = 0$

$\therefore \tan x = -1.$

$\therefore x = \frac{3\pi}{4}, \frac{-\pi}{4}, \frac{7\pi}{4} \ldots$

$g''(x) = -\sin x + \cos x.$

$g''(-\frac{\pi}{4}) = +\frac{1}{\sqrt{2}} + 1/\sqrt{2} = \bullet$ +ve $> 0. \Rightarrow$ min.

$g''(\frac{3\pi}{4}) = -1/\sqrt{2} + (-1/\sqrt{2}) < 0 \Rightarrow$ max.

$\therefore$ max. $f(\frac{3\pi}{4}) = e^{\sin(\frac{3\pi}{4}) - \cos\frac{3\pi}{4}}$

$= e^{1/\sqrt{2} - (-1/\sqrt{2})}$

$= e^{\sqrt{2}}.$

GATE-2012

Ex-3 The max. value of the f^n $f(x) = x^3 - 9x^2 + 24x + 5$. in $[1, 6]$. is____

Ans: (a) 21 (b) 25

(b) 41 (d) 46.

$\rightarrow f'(x) = 3x^2 - 18x + 24 = 0$

$x^2 - 6x + 8 = 0$

$\therefore\ f''(x) = 6x - 18.$

$\therefore\ f''(2) = 12 - 18 = -6 < 0$ max at $x = 2.$

$f''(4) = 24 - 18 = +6 > 0$ min at $x = 4.$

$\therefore\ f(2) = 8 - 36 + 48 + 5 = 25$

$f(1) = 21.$

$f(8) = 41.$

Ex-4 if $y = a\log|x| + bx^2 - x$ has extreme values at $x = 4/3$ and $x = -2$, then the values of $a, \& \ b$ are —?

Ans:

$$y' = \frac{a}{\cancel{|x|}} \times \frac{\cancel{|x|}}{x} + 2bx - 1.$$

$$y' = \frac{a}{x} + 2bx - 1 = 0.$$

$$\therefore\ 2bx^2 - x + a = 0.$$

$$\therefore\ x = -2, \quad x = 4/3.$$

$$\therefore\ 8b + 2 + a = 0$$

$$a + 8b = -2. \quad — (1)$$

$$\therefore\ 2 \times b \times \frac{16}{9} - \frac{4}{3} + a = 0$$

$$\therefore\ 32b - 4 + 3a = 0$$

$$\therefore\ 3a + 32b = 4. \quad — (2)$$

(or)

$$\therefore\ \left(x - \frac{4}{3}\right)(x + 2) = 0.$$

$$(3x - 4)(x + 2) = 0$$

$$3x^2 + 6x - 4x - 8 = 0$$

$$\therefore\ -3/2\, x^2 - 2x + 4 = 0.$$

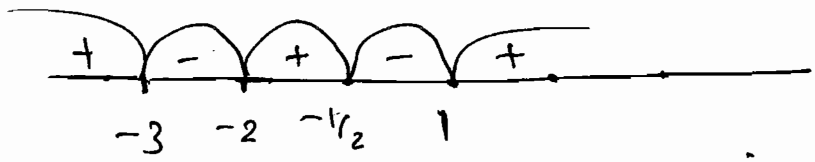
$$\therefore\ 2b = -3/2, \quad \boxed{a = 4.}$$

$$\therefore\ \boxed{b = -3/4}$$

Ex - 5 If $f(X) = (x-1) ($ ...

$f(x)$ at $x = -1/2$ is ______.

(a) max (b) min (c) no extrem (d) non. of the



→ Third term of $f''(x)$ is

$(x-1)(x+2)^2(x+3)2$

at $x = -1/2$ $\quad - \quad + \quad +$

so, $f''(x) < 0 \Rightarrow$ max.

* Maxima and Minima of ... of two variables:

→ Let, $f(x,y)$

Let, $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$.

$$s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}.$$

⇒ **Method:**

① find p, q, r, s, t.

② Equate p & q to zero for obtaining stationary points

③ At each stationary points find r, s, t.

ⓐ If $rt - s^2 > 0$ & $r > 0 \rightarrow$ min.

ⓑ If $rt - s^2 > 0$ & $r < 0 \rightarrow$ max.

ⓒ If $rt - s^2 < 0$ then $f(x,y)$ has no extreme at that stationary points. and such points are called saddle points.

Ex-1 $f(x,y) = 1 - x^2 - y^2$ has

(a) max at $(0,0)$ (b) min at $(0,0)$

(c) $(0,0)$ as a saddle point

(d) None

Ans: $p = \frac{\partial f}{\partial x} = -2x$

$q = \frac{\partial f}{\partial y} = -2y.$

$r = \frac{\partial^2 f}{\partial x^2} = -2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0.$

$t = \frac{\partial^2 f}{\partial y^2} = -2.$

$\left.\begin{array}{l} p=0 \Rightarrow x=0 \\ q=0 \Rightarrow y=0 \end{array}\right\}$ $(0,0)$ is st. point.

$\therefore rt - s^2 = 4 - 0 > 0$

Now, $r = -2 < 0 \Rightarrow$ max.

Ex-2 $f(x,y) = x^2y + xy^2 - xy$ has min value at

(a) $(0,0)$ (b) $(1/3, 1/3)$ (c) $(-1/3, -1/3)$ (d) None.

Ans. $p = 2xy + y^2 - y.$

$q = x^2 + 2xy - x$

$r = 2y.$

$s = 2x + 2y - 1.$

$t = 2\cancel{x}.$

Now, $p=0$

$y^2 + 2yx - y = 0$

$\therefore \quad (y^2 - x^2) - (y + x) = 0$

$\therefore \quad (y - x)\,[\,y + x - 1\,] = 0$

$\boxed{x = y}$, $\quad x + y = 1$

$\cancel{2y = 1}$

$\cancel{\boxed{x = 1/2}} \qquad \cancel{\boxed{y = 1/2}}$

$\therefore \quad y^2 + 2y^2 - y = 0$

$3y^2 - y = 0$

$y = 0, \quad y = 1/3$

$x = 0, \quad x = 1/3.$

$(0,0)$ and $(1/3, 1/3)$ is st. pt.

At $(0,0)$ $\quad r = 0, \quad \cancel{t} = 0, \quad s = -1, \quad t = 0.$

$rt - s^2 = -1 < 0.$

No extrime

At $(1/3, 1/3)$ $\quad r = 2/3, \quad s = 1/3, \quad t = 2/3.$

$\therefore \quad rt - s^2 = \frac{4}{9} - 1/9 = 1/3 > 0.$

$\therefore$ Now, $r = 2/3 > 0$ so min. at $(1/3, 1/3)$.

Ex-3 The maximum value of the fn $f(x,y) = x^3 + y^3 + 3xy$ is ___ ?

Ans:

$p = 3x^2 + 3y.$

$q = 3y^2 + 3x.$

$r = 6x$

$s = 3$

$t = 6y.$

$\therefore \quad \cancel{rt} - s^2$

$p = 0$

so, $3x^2 + 3y = 0$

$q = 0$

$3y^2 + 3x = 0$

$3(x^2 - y^2) + 3(x - y) = 0$

$\therefore \quad \boxed{x = y} \qquad \boxed{x + y = -1}$

$\therefore \; 3x \ldots$

$x = 0, -1$

$y = 0, -1$

$\therefore \; (0,0), \; (-1,-1)$ are st. pt.

Now at $(0,0)$ $\quad r = 0, \quad s = 3, \quad t = 0$

$\therefore \; rt - s^2 = 0 - 9 < 0 \rightarrow$ No extreme

at $(-1,-1)$ $\quad r = -6, \quad s = 3, \quad t = -6.$

$\therefore \; 36 - 9 = 27 > 0$ so,

$r = < -6 \Rightarrow$ ~~maxi~~ max at $(-1,-1)$.

$\therefore \; f(-1,-1) = -1 - 1 + 3 = 1.$

Ex-2 A Rectangular box open at the top then is to have a volume of 32 c-ft such that the dimension of the box it construction the material required for are —?

ⓐ 4, 4, 2 ⓑ 8, 8, 1/2, ⓒ 2, 2, 8 ⓓ 16, 1, 2.

Ans:

$S = xy + 2yz + 2xz \quad$ ($\because$ 5 face, top is open).

$V = xyz, = 32$

$\therefore \; S = xy + \frac{64}{x} + \frac{64}{y}.$

$\therefore \; p = y - \frac{64}{x^2} = 0$

$q = x - \frac{64}{y^2} = 0$

$\therefore\ y = \dfrac{64 \times 8}{(64)^{\pi}}$

$\therefore\ y^3 = 64$

$\therefore\ \boxed{y = 4}$

$\therefore\ \boxed{x = 4}$

$\therefore$ $(4,4)$ is stationary point.

Sol Ans (a) $(4, 4, 2)$.

Ex-5 The distance betn origin and a point nearest to it on the surface $z^2 = 1+xy$ is ____.

(a) 1 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) $\sqrt{3}/2$.

Ans: Let $P(x,y,z)$ be a pt. on $z^2 = 1+xy$.

$D = OP = \sqrt{x^2+y^2+z^2}$.

$\therefore\ D = OP = \sqrt{x^2+y^2+1+xy}$.

Let, $f(x,y) = x^2+y^2+1+xy$.

$\therefore\ p = 2x + y = 0$ $\quad y = -2x$

$q = 2y + x = 0$ $\quad x = -2y$.

$x = -2(-2x)$

$4x - x = 0$

$x = 0, y = 0$

$\therefore\ (0,0)$ is st. pt.

$\therefore$ ~~at $(0,0)$ $z = 1+0 = 1$~~ $\quad rt - s^2 = 3 > 0$, $r = 2 > 0$ min. $\quad z^2 = 1+0.0$

$\boxed{z = \pm 1}$

$(0,0,1)$

$(0,0,-1)$.

$\therefore\ D = OO = \sqrt{1+0+0} = 1$.

$\boxed{D = 1}$

* Constrained maxima and minima:

⇒ Langrange's method of undetermined multipliers:

→ Let $f(x,y,z)$ where $\phi(x,y,z) = c$ —①

Consider $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$.

$F_x = 0,\quad F_y = 0,\quad F_z = 0.$

$$\therefore \left.\boxed{\begin{aligned} &\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0. \quad —② \\ &\frac{\partial f}{\partial y} + \lambda \cdot \frac{\partial \phi}{\partial y} = 0. \quad —③ \\ &\frac{\partial f}{\partial z} + \lambda \cdot \frac{\partial \phi}{\partial z} = 0 \quad —④ \end{aligned}}\right\} \text{Langrange's eq}^n.$$

Solving eq^n ① to ④ we obtained the value's of x, y, z, λ.

→ (x,y,z) is called st. pt. and $f(x,y,z)$ is called extreme value.

Ex-1 The value of the f^n $x^2+y^2+z^2$, $x+y+z=1$ is ___ ?

Ans: $f = x^2+y^2+z^2$, $\phi = x+y+z-1$.

$$\therefore \left.\begin{aligned} &2x + \lambda(1) = 0. \Rightarrow -\tfrac{\lambda}{2} = x \\ &2y + \lambda(1) = 0. \Rightarrow -\tfrac{\lambda}{2} = y \\ &2z + \lambda(1) = 0. \Rightarrow -\lambda/2 = z. \end{aligned}\right\} x = y = z = 1/3.$$

$= \quad -\lambda/2 + (-\lambda/2) - \lambda/2 = 1.$ $\quad \boxed{\lambda = 2/3}$

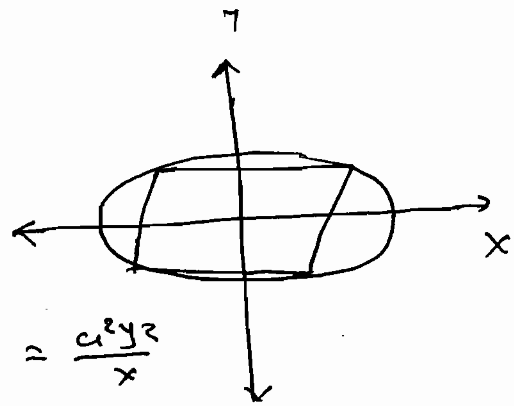
Extreme value $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = 1/3.$

Ex-2 The volume of greatest Parallelopiped in the Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$ is — ?

(a) $\frac{abc}{3\sqrt{3}}$ (b) $\frac{abc}{\sqrt{3}}$

(c) $\frac{8abc}{3\sqrt{3}}$ (d) $\frac{27abc}{\sqrt{3}}$.

Ans: Let. $P(2x, 2y, 2z)$.



$\therefore V = 8xyz.$

$\therefore 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow -\frac{2\lambda}{8} = \frac{a^2yz}{x}$

$8xz + \lambda\left(\frac{2y}{b^2}\right) = 0 \Rightarrow -\frac{2\lambda}{8} = \frac{b^2xz}{y}$

$8xy + \lambda\left(\frac{2z}{c^2}\right) = 0. \Rightarrow -\frac{2\lambda}{8} = \frac{c^2xy}{z}.$

$\therefore \frac{a^2yz}{x} = \frac{b^2xz}{y}$

$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2}.$

Similarly, $\frac{y^2}{b^2} = \frac{z^2}{c^2}.$

$\therefore 3\frac{x^2}{a^2} = 1.$

$x = \frac{a}{\sqrt{3}}.$, $y = a/\sqrt{3}$, $z = a/\sqrt{3}.$

St. Pt. is $\left(\frac{a}{\sqrt{3}}, b/\sqrt{3}, c/\sqrt{3}\right).$

Extreme value $\therefore f\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}.$

Multiple Integrals:

* Double Integral:

→ $f(x,y) \longrightarrow R$

$\delta R_1, \delta R_2, \cdots \delta R_n$.

$(x_i, y_i) \rightarrow \delta R_i$

→ $$\lim_{n \to \infty} \left[\sum_{i=1}^{n} \{ f(x_i, y_i)\, \delta R_i \} \right] = \iint_R f(x,y)\, dx\, dy.$$

∴ Area $A = \iint_R 1 \, dx\, dy$.

→ Let, $f(x,y)$ be defined at each point of a region R. devide the region R into n sub regions each of area $\delta R_1, \delta R_2, \cdots \delta R_n$

Let, (x,y) be an arbitury point in a sub region with area δR_i. Then.

$$\lim_{n \to \infty} \left[\sum_{i=1}^{n} \{ f(x,y)\, \delta R_i \} \right] = \iint_R f(x,y)\, dx\, dy.$$

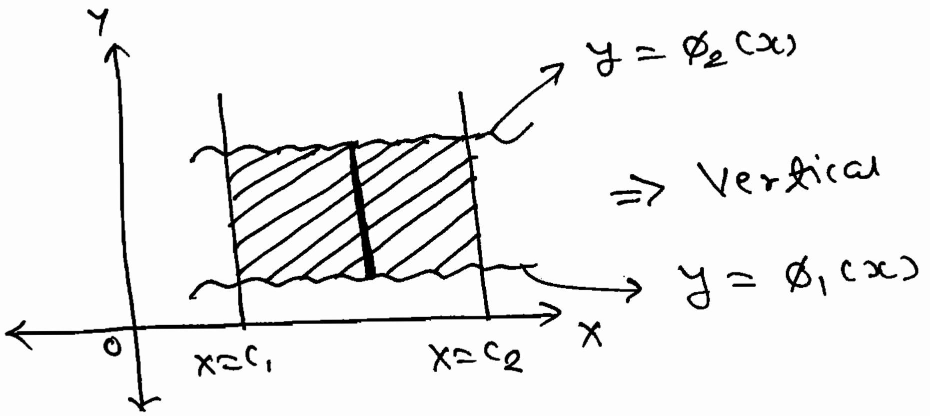
* Method of Evaluation

⊙ Case-(i):

→ When the limits are

$y = \phi_1(x), \quad y_2 = \phi_2(x)$

$x = c_1 \quad \& \quad x = c_2$

→ 

⇒ Vertical Strip

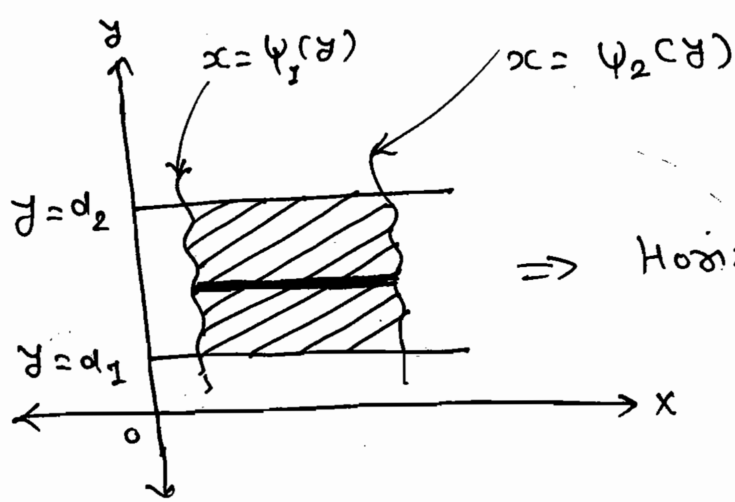
→ $$\iint\limits_R f(x,y)\,dx\,dy = \int_{x=c_1}^{x=c_2}\left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\right]dx.$$

⊙ Case - 2:

→ When the limits are

$x = \psi_1(y) \quad , \quad x = \psi_2(y)$

$y = d_1 \quad , \quad y = d_2$



⇒ Horizontal Strips.

$$\rightarrow \iint_R f(x,y)\,dx\,dy$$

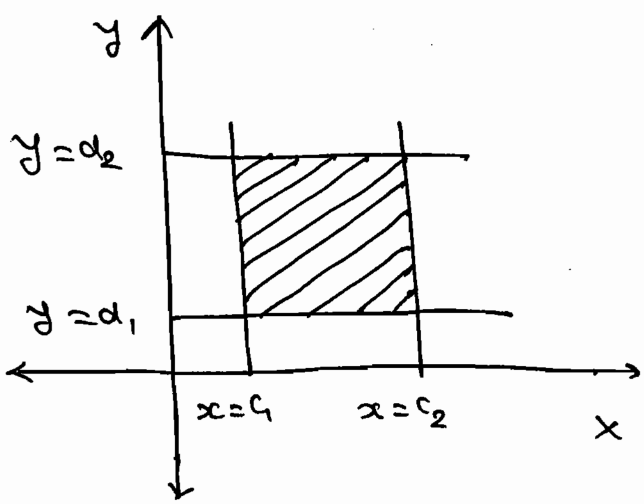
$$= \int_{y=d_1}^{y=d_2} \left[\int_{x=\psi_1(y)}^{x=\psi_2(y)} f(x,y)\,dx \right] dy$$

case (iii):-

$\rightsquigarrow$ when the limits are

$y = d_1$ to $y = d_2$

$x = c_1$ to $x = c_2$.



$$\iint_R f(x,y)\,dx\,dy$$

$$= \int_{x=c_1}^{x=c_2} \left[\int_{y=d_1}^{y=d_2} f(x,y)\,dy \right] dx = \int_{y=d_1}^{y=d_2} \left[\int_{x=c_1}^{x=c_2} f(x,y)\,dx \right] dy$$

Ex-1 Evaluate the following

① $\int_{1}^{2}\int_{3}^{4} \frac{1}{(x+4)^2}\, dx\, dy =$ ______.

Ans:

$$I = \int_{1}^{2}\int_{3}^{4} \frac{1}{(x+4)^2}\, dx \cdot dy$$

$$= \int_{1}^{2} \left[\left\{-\frac{1}{x+4}\right\}_3^4 \right] dy.$$

$$= \int_{1}^{2} \left(\frac{1}{y+3} - \frac{1}{y+4} \right) dy.$$

$$= \left[\log\left(\frac{y+3}{y+4}\right) \right]_1^2$$

$$= \log \frac{5}{6} - \log\left(\frac{4}{5}\right).$$

$$= \log\left(\frac{5}{6} \times \frac{5}{4}\right)$$

$$I = \log\left(\frac{25}{24}\right).$$

② $\int_{0}^{3}\int_{0}^{x} (6-x-y)\, dy\, dx =$ ______.

Ans:

$$I = \int_{0}^{3} \left[6y - xy - \frac{y^2}{2} \right]_0^x \cdot dx$$

$$= \int_{0}^{3} \left(6x - x^2 - \frac{x^2}{2} \right) dx.$$

$$I = \left[3x^2 - \frac{x^3}{3} - \frac{x^3}{6}\right]_0$$

$$\therefore I = 27 - 9 - \frac{9}{2}.$$

$$= 18 - \frac{9}{2}$$

$$\therefore \boxed{I = \frac{27}{2}}$$

③ $\int_0^4 \int_0^{y^2} e^{x/y}\, dx\, dy =$ ______.

Ans: $I = \int_0^4 \left[\frac{e^{x/y}}{1/y}\right]_0^{y^2} dy.$

$$= \int_0^4 y \cdot [e^y - 1] \cdot dy.$$

$$= \int_0^4 (ye^y - y)\, dy.$$

$$= \left[y \cdot e^y - e^y - \frac{y^2}{2}\right]_0^4.$$

$$= 4e^4 - e^4 - 8 + 1$$

$$\therefore I = 4e^4 - e^4 - 9 + 2$$

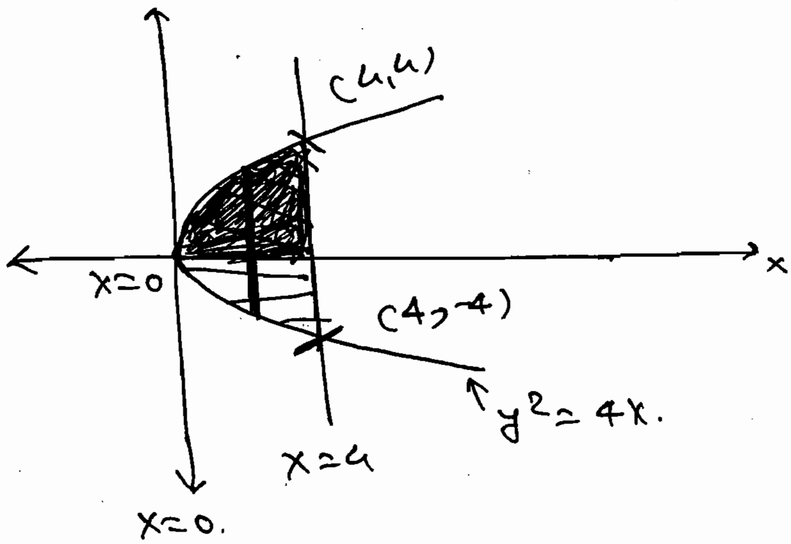
$$= 4e^4 - e^4 - 7$$

$$\therefore \boxed{I = 3e^4 - 7}$$

(4) $\iint\limits_R xy \, dx \, dy$ where, R

bounded by $y^2 = 4x$, $x = 4$ in the 1st quadrant is ___.

Ans: $y^2 = 4x$.



① if vertical

$y = 0$ to $y = 2\sqrt{x}$.

$x = 0$ to $x = 4$.

② if horizontal

$x = 0$ to $x = \frac{y^2}{4}$

$y = 0$ to $y = 4$.

$x = 4$. $y^2 = 16$

$\therefore y = \pm 4$.

$\therefore$ (0, 0) to (4, 4).

$$I = \int_{x=0}^{x=4} \int_{y=0}^{y=2\sqrt{x}} xy \, dy \, dx.$$

$$\therefore I = \int_0^4 \int_0^{2\sqrt{x}} x \cdot 4 \frac{y^3}{4} \, dy \, dx.$$

$$= \int_0^4 \left[\frac{y^4}{16} \right]_0^{2\sqrt{x}}$$

$$= \int_0^4 \frac{16 x^2}{16} \, dx = \frac{x^3}{3} = \frac{64}{3}.$$

(5) $\iint_R r^2 \sin\theta \cdot dr\, d\theta$ where R is ... above the initial line.

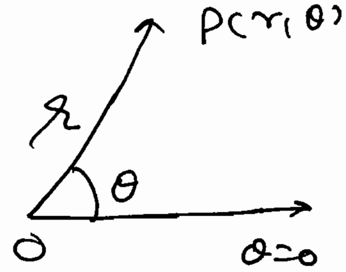
Ans: $R \rightarrow$ semi circle $r = 2a\cos\theta$.

NOTE:

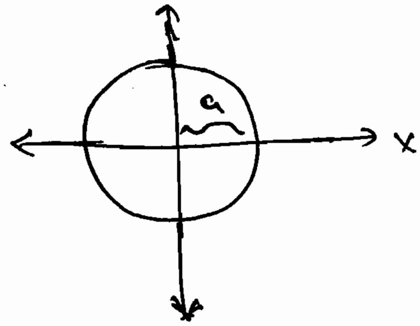
$x = r\cos\theta$

$y = r\sin\theta$

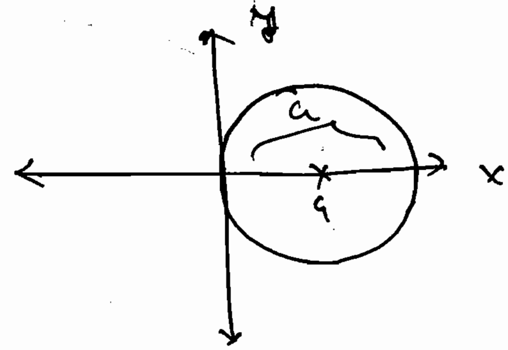
$\Rightarrow x^2 + y^2 = r^2$



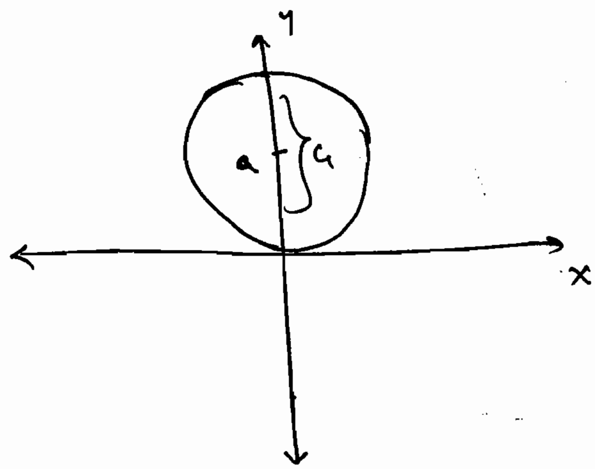
(1) $r = a$



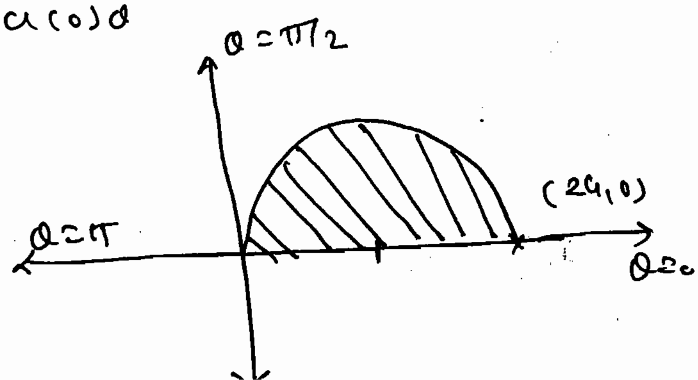
(2) $r = a\cos\theta$



(3) $r = a\sin\theta$



Now, $r = 2a\cos\theta$



$r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\pi/2$.

$$I = \int_0 \int_0 r^2 . \sin\theta . dr d\theta .$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} . d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{8a^3 \cos^3\theta}{3}\right] . d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \cos^3\theta . \sin\theta . d\theta$$

$$= \frac{8a^3}{3} \left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{8a^3}{3} \left[0 + \frac{1}{4}\right]$$

$$\therefore \boxed{I = \frac{2a^3}{3}.}$$

* Area of Region:

→ The Area of the region bounded by the curves $y = f(x)$ & $y = g(x)$ betn $x = x_1$ & $x = x_2$ is

$$A = \int_{x_1}^{x_2} \int_{f(x)}^{g(x)} 1 . dy\, dx .$$

(OR)

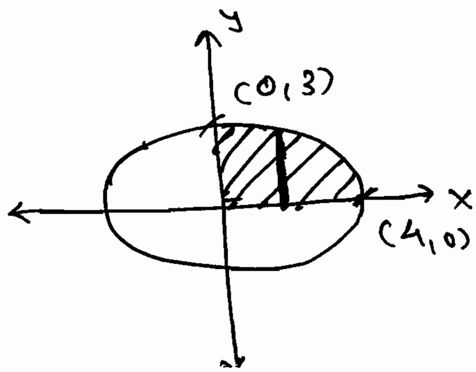
$$A = \int_{x_1} \left(f(x) - g(x) \right) dx.$$

→ In Polar form,

$$A = \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} r \, dr \, d\theta .$$

Ex - 1 The area bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is ______.

Ans:



Vertical strip,

$x = 0$ to $x = 4$

$y = 0$ to

$y = \frac{3}{4}\sqrt{16-x^2}.$

$$I = 4 \int_0^4 \int_0^{\frac{3}{4}\sqrt{16-x^2}} dy \, . \, dx$$

$$= 4 \int_0^4 \frac{3}{4} \sqrt{16-x^2} \, . \, dx.$$

$$\therefore I = 3 \left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2} \sin^{-1} \frac{x}{4} \right]_0^4.$$

$$= 3 \left[8 \times \frac{\pi}{2} \right]$$

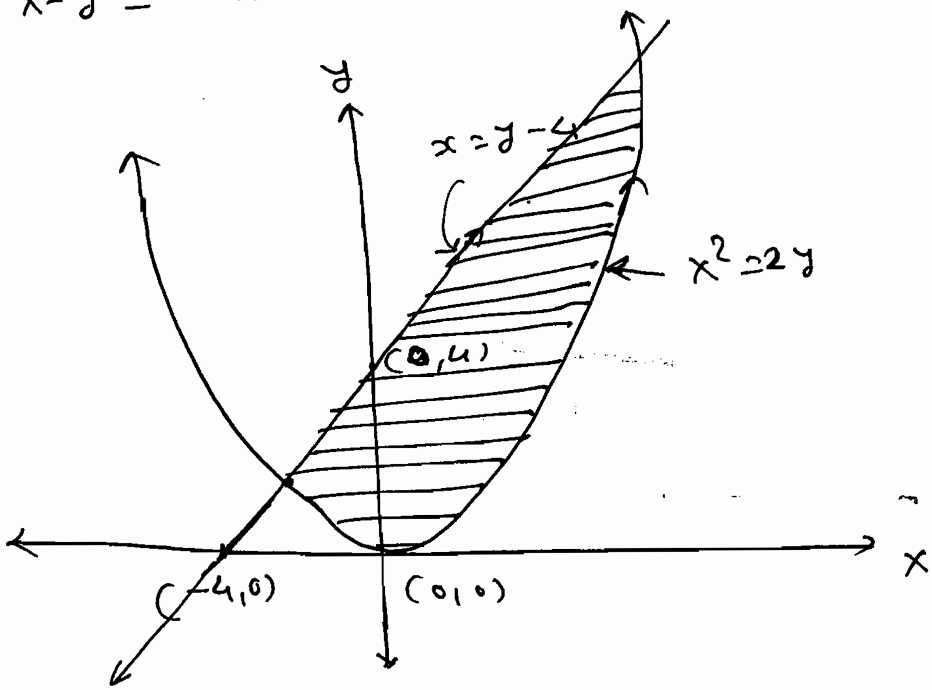
$$\boxed{I = 12\pi}$$

$$\boxed{A_e = \pi ab.}$$

The Area [illegible]
$2y = x^2$ and the line $x = y - 4$ is ___.

(a) 6 (b) 18 (c) ∞ (d) non of them.

Ans:
$$x^2 = 2y.$$
$$x - y = -4.$$



$x = -2$ to $x = 4$.

$y = x + 4$.

$y = x^2/2$

$$\therefore \quad x^2 = 2(x+4).$$
$$x^2 - 2x - 8 = 0.$$
$$x = -2, 4.$$
$$\therefore \quad y = 2, 8.$$

$$\therefore \quad I = \int_{-2}^{4} \int_{x^2/2}^{x+4} dy\, dx$$
$$= \int_{-2}^{4} (x^2/2 - x - 4)\, dx$$
$$= \left[\frac{x^3}{6} - \frac{x^2}{2} - 4x \right]_{-2}^{4}$$
$$= \left(8 + 16 - \frac{64}{6}\right) - \left(2 - 8 + 8/6\right)$$

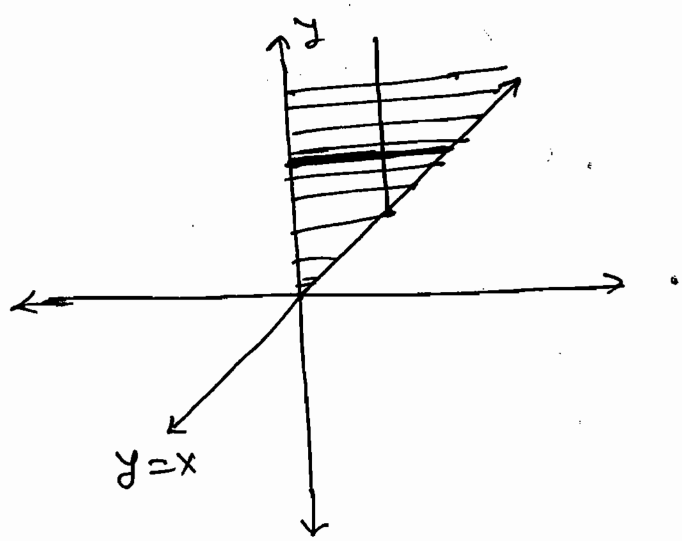
☆ Change of order of integration

→ **Ex-1** The value of $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx =$ ______.

Ans: Given limits are

$y = x$ to $y = \infty$.

$x = 0$ to $x = \infty$



Horizontal strip

$x = 0$ to $x = y$

$y = 0$ to $y = \infty$

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy$$

$$\therefore I = \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y} \, dy.$$

$$= \left[-e^{-y} \right]_0^\infty$$

$$= 0 + 1.$$

$$\therefore \boxed{I = 1}$$

(2) By reversing the ... in a double integral $\int_0^2 \int_{y^3}^{4\sqrt{2y}} f(x,y)\,dx\,dy$.

it may be represented as $\int_p^q \int_r^s f(x,y)\,dy\,dx$.

then the value of $q \times r =$ ____.

(a) $\frac{x^2}{2}$ (b) $\frac{x^2}{4}$ ✓ (c) $x^{1/3}$ (d) 0.

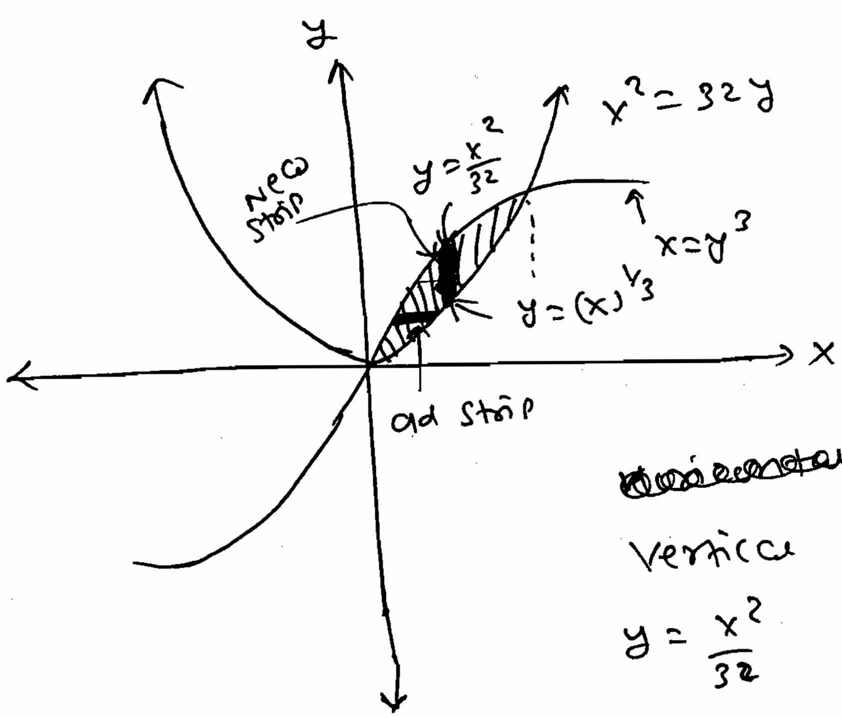
Ans: Given limits are.

$y = 0$ to $y = 2$.

$\therefore$ $x = y^3$ to $x = 4\sqrt{2y}$.

$x^2 = 16\ 2y$.

$x^2 = 32y$.



~~horizontal~~

Vertical strip

$y = \frac{x^2}{32}$ to $y = (x)^{1/3}$

$x = 0$ to $x = 8$.

$$\int_0^8 \int_{x^2/32}^{x^{1/3}} f(x,y)\,dy\,.\,dx$$

$q = 8$

$r = \frac{x^2}{32}$

$q \times r = 8 \times \frac{x^2}{32}$

$= x^2/4$.

* Triple Integration:

$\rightarrow$ $\phi(x, y, z) \longrightarrow R$

$\delta v_1, \delta v_2, \ldots, \delta v_n.$

$(x_i, y_i, z_i) \longrightarrow \delta v_i$

$$\lim_{n \to \infty} \left[\sum_{i=1}^{n} \left[\phi(x_i, y_i, z_i)\, \delta v_i \right] \right]$$

$$= \iiint_R \phi(x, y, z)\, dx\, dy\, dz.$$

Let $z = f_1(x, y)$ to $f_2(x, y)$

$y = g_1(x)$ to $g_2(x).$

$x = c_1$ to $x = c_2.$

Then

$$\boxed{\iiint_R \phi(x, y, z)\, dz\, dy\, dx = \int_{x=c_1}^{c_2} \left[\int_{g_1(x)}^{g_2(x)} \left\{ \int_{f_1(x,y)}^{f_2(x,y)} \phi(x, y, z)\, dz \right\} dy \right] dx.}$$

① $$\int_0^1 \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \cdot dy \cdot dx.$$

Ans:

$$I = \int_0^1 \int_0^x \left[e^{x+y+z} \right]_0^{x+y} dy \cdot dx.$$

$$= \int_0^1 \int_0^x \left[e^{2x+2y} - e^{x+y} \right] dy \cdot dx.$$

$$= \int_0^1 \left[\frac{e^{2x+2y}}{2} - e^{x+y} \right]_0^x dx.$$

$$= \int_0^1 \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right] dx$$

$$= \int_0^1 \frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \, . dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x \right]_0^1$$

$$= \frac{e^4}{8} - \frac{3e^2}{4} + e - \frac{1}{8} + 3/4 + 1.$$

$$= \frac{e^4 - 6e^2 + 8e - 1 + 6 + 8}{8}$$

$$\therefore \boxed{I = \frac{e^4 - 6e^2 + 8e + 13}{8}}$$

② The value of $\iiint_R y \, dx\,dy\,dz$ where, R is the region bounded by the planes $x=0$, $y=0$, $z=0$
$x+y+z=1$.

$\therefore$ Put $y=0$ & $z=0$

$\therefore$ $x=1$

So, $\rightarrow x=0$ to $x=1$

$\rightarrow y=0$ to $y=1-\cancel{x}$

$\rightarrow z=0$ to $z=1-x-y$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \, dz \, dy \, dx.$$

$$= \int_0^1 \int_0^{1-x} [yz]_0^{1-x-y} \, . \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} (y - yx - y^2) \, . \, dy \, dx.$$

$$= \int_0^1 \left(\frac{y^2}{2} - \frac{y^2 x}{2} - \frac{y^3}{3}\right)_0^{1-x} . \, dx$$

$$= \int_0^1 \frac{(1-x)^2}{2} - \frac{x(1-x)^2}{2} - \frac{(1-x)^3}{3} . \, dx.$$

$$= \int_0^1 \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \, dx$$

$= \frac{1}{6} \int_0 (1-x)^3 \cdot dx.$

$= \frac{1}{6} \left[\frac{(1-x)^4}{-4} \right]_0^1$

$= \frac{1}{6} \left[0 + \frac{1}{4} \right].$

$= \frac{1}{24}.$

* Change of Variables in Double Integral:

→ Let, $x = f(u,v)$, $y = g(u,v)$.

Then $\iint_R \phi(x,y)\, dx\, dy.$

$= \iint_R \phi(f,g)\, |J|\, du\, dv.$

$= \iint_R \psi(u,v)\, |J|\, du\, dv.$

$|J| \rightarrow$ Jacobian Transformation.

$\therefore \boxed{|J| = J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$

$$(x, y) \rightarrow (r, \theta).$$

$$\therefore \boxed{x = r\cos\theta}, \quad \boxed{y = r\sin\theta.}$$

$$\Rightarrow \boxed{x^2 + y^2 = r^2.}$$

$$|J| = J\left[\frac{x, y}{u, v}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r.$$

$$\boxed{|J| = r}$$

$$\boxed{\iint_R \phi(x, y)\, dx\, dy = \iint_R \phi(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.}$$

* In Triple Integral:

① Cartesian form → Cylindrical form.
Polar ↓ (above "form")

$$(x, y, z) \longrightarrow (r, \theta, z).$$

$$\boxed{x = r\cos\theta}, \quad \boxed{y = r\sin\theta}, \quad \boxed{z = z.}$$

$$\Rightarrow \boxed{x^2 + y^2 = r^2, \quad z = z.}$$

$$|J| = J\left[\frac{x, y, z}{r, \theta, z}\right] = r. \qquad \boxed{|J| = r}$$

$$\therefore \boxed{\iiint_R \phi(x, y, z)\, dx\, dy\, dz = \iiint_R \phi(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz}$$

(e) Cartesian form $\longrightarrow$ Spherical form

$(x, y, z) \longrightarrow (r, \phi, \theta).$

$\boxed{x = r\sin\theta \cdot \cos\phi}$, $\boxed{y = r\sin\theta \cdot \sin\phi}$, $\boxed{z = r\cos\theta.}$

$\rightarrow$ $x^2 + y^2 + z^2 = r^2$, $|J| = J\left[\frac{x, y, z}{r, \phi, \theta}\right] = r^2\sin\theta.$

$\boxed{|J| = r^2\sin\theta.}$

* Cylinder

$r = 0$ to R

$\theta = 0$ to 2π

$z = z_1$ to z_2

Sphere

$r = 0$ to R

$\phi = 0$ to 2π

$\theta = 0$ to π.

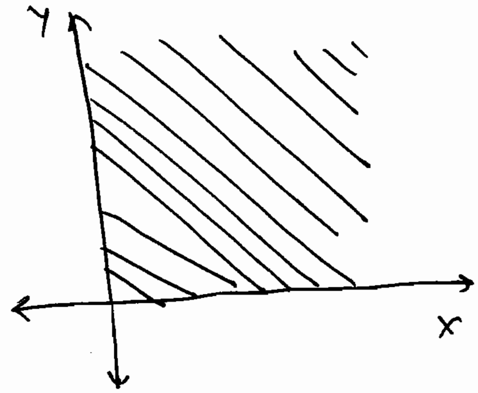
Ex - 1

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy = \_\_\_\_\_\_$$

Ans:

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

$$x^2+y^2 = r^2. \quad , \quad |J| = r.$$



$$r = 0 \text{ to } r = \infty$$
$$\theta = 0 \text{ to } \theta = \pi/2.$$

$$I = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} \cdot r \, dr\, d\theta$$

$$r^2 = t$$
$$2r \cdot dr = dt$$
$$r\,dr = \frac{dt}{2}$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-t} \cdot dt \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1}\right]_0^\infty \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} [0+1]\, d\theta.$$

$$= \frac{1}{2} \times [\theta]_0^{\pi/2}$$

$$\therefore \boxed{I = \frac{\pi}{4}}$$

Ex-: By changing to polar co-ordinated the

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dx\,dy\,dz$$ may be

represented as.

Let, $x = r\sin\theta.\cos\phi$

$y = r\sin\theta.\sin\phi$

$z = r\cos\theta.$

$x^2+y^2+z^2 = r^2.$

$|J| = r^2 \sin\theta.$

$z = 0$ to $z = \sqrt{1-x^2-y^2}.$

Region is +ve octant of sphere.

$r = 0$ to 1

$\phi = 0$ to $\pi/2.$

$\theta = 0$ to $\pi/2.$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1}{\sqrt{1-r^2}} . r^2 \sin\theta \; dr\,d\phi\,d\theta.$$

Ex-③ By the change of variables $x(u,v) = uv$, $y(u,v) = v/u$ in a double integral the integrade $f(x,y)$ changes to $f(uv, v/u)\,\phi(u,v)$. then $\phi(u,v) =$ ____.

$f(x,y) \longrightarrow f(uv, v/u)\,\phi(u,v)$

(A) $2v/u$ (B) v/u.

(C) $2uv$ (d) 1.

Ans: $x(u,v) = uv,\quad y =$

$\therefore \quad |J| = J\frac{(x,y)}{(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \phi(x,u)$

$$= \begin{vmatrix} v & u \\ -v/u^2 & 1/u \end{vmatrix}$$

$$= v/u + v/u^2 \cdot u$$

$$= \frac{v}{u} + v/u$$

$$\boxed{\phi(u,v) = \frac{2v}{u}.}$$

☆ Length of curve:

⇝ The length of an arc of a curve $y = f(x)$ b/w $x = x_1$ and $x = x_2$ is

$$\boxed{L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$ ✓

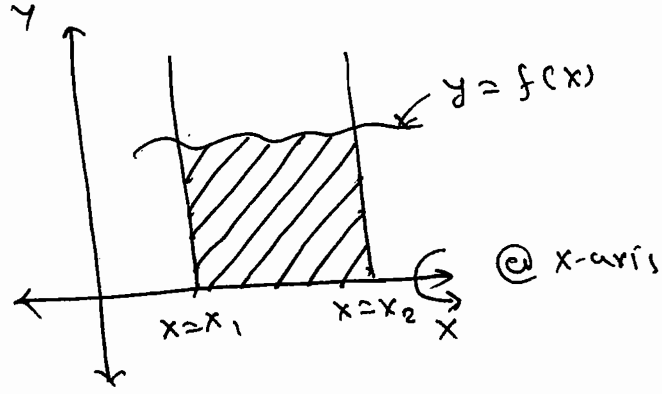
⇝ In Polar form::

$$\boxed{L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.}$$

Volume of Solid of Revolution

① The Volume of Solid generated by revolving the area bounded by the curve $y = f(x)$ b/w $x = x_1$ and $x = x_2$ about x-axis is integral x_1 to x_2

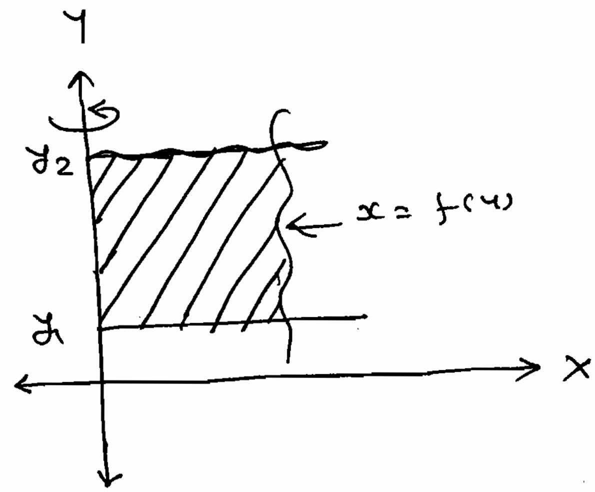
$$\therefore \boxed{V = \int_{x_1}^{x_2} \pi y^2 \, dx}$$



② about y-axis:

$$V = \int \pi \cancel{x}^2 \, dy$$

$$\boxed{V = \int_{y_1}^{y_2} \pi x^2 \, dy}$$



* In polar form:

① About initial line $\theta = 0$.

$$\boxed{V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta \cdot d\theta}$$

② About the line $\theta = \pi/2$

$$V = \int^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta \cdot d\theta.$$

Ex-1 The Length of the curve $y = \frac{2}{3}x^{3/2}$ b/w $x=0$ and $x=1$ is ______.

(a) 0.27 (b) e (c) 1 (d) 1.22

Ans:

$$y = \frac{2}{3} x^{3/2}$$

$$\therefore L = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx.$$

$$\therefore \frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{+\frac{1}{2}} = \frac{\cancel{4}}{\cancel{4}} \times x^{+1/2} = \sqrt{x}.$$

$$\therefore L = \int_0^1 \sqrt{1+x} \cdot dx$$

$$= \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{2}{3} \times \left[2\sqrt{2}-1\right].$$

$$\therefore \boxed{L = 1.22}$$

Ex-2 The Length of the curve $y = \log(\sec x)$ bet^n $x=0$ and $x=\frac{\pi}{4}$ is ______.

Ans:

$$y = \log(\sec x).$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \cdot \tan x = \tan x.$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x} \cdot dx$$

$$= \int^{\pi/4} \sec x \cdot dx.$$

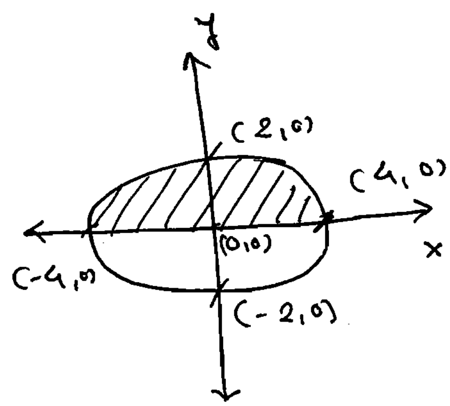
$$I = \Big[\log|\sec x + \tan x|\Big]_0$$

$$\therefore \boxed{L = \log(\sqrt{2}+1)}.$$

Ex-3 The volume of solid generated by revolving the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ @ x axis is ____ .

Ans: @ x axis

$$V = \int_{x_1}^{x_2} \pi y^2 \cdot dx.$$



$$y^2 = \frac{4}{16}\cancel{\sqrt{x^2-16}}\ \sqrt{16-x^2}$$

$$\therefore y^2 = \frac{1}{4}\cancel{\sqrt{x^2-16}}.\ \sqrt{16-x^2}$$

$$\therefore V = \int_{-4}^{4} \pi \times \frac{1}{4}\left(\sqrt{16-x^2}\right)^2 . dx.$$

$$V = 2\times\frac{\pi}{4}\int_0^4 16-x^2 \ . dx.$$

$$= \frac{\pi}{2}\times\left[16x - \frac{x^3}{3}\right]_0^4.$$

$$= \frac{\pi}{2}\times\left[\frac{2}{3}\times 64\right]$$

$$\therefore V = \frac{64\pi}{3}.$$

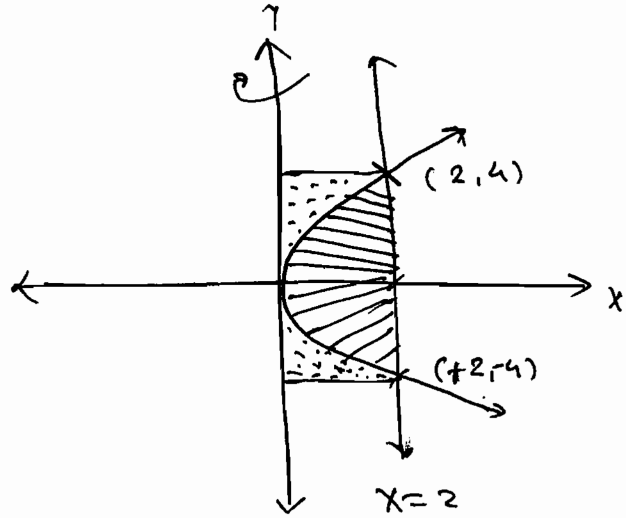
Ex-4 $y^2 = 8x$ and $x = \ldots$
is ______.

Ans: (A) $\frac{128\pi}{5}$

(B) $\frac{128}{5\pi}$

(C) $\frac{127\pi}{5}$

(D) None



→ The Volume generated by revolving the area bounded by the straight line $x=2$ b/w $y=-4$ & $y=+4$. @ y-axis. is

$$V_1 = \int_{-4}^{4} \pi x^2 . dy. \qquad x=2$$

$$\therefore V_1 = 2\pi \int_0^4 4.dy$$

$$= 8\pi \times [4].$$

$$\therefore V_1 = 32\pi$$

→ similarly, the Volume generated by revolving the area bounded by the parabola $y^2=8x$ b/w ~~x=0~~ $y=-4$ & $y=+4$. is

$$V_2 = \int_{-4}^{+4} \pi x^2 \, dy.$$

$$\therefore V_2 = 2 \int^4 \pi \frac{y^4}{64} . dy$$

$$\therefore V_2 = \frac{\pi}{32}\left[\frac{y^5}{5}\right]_0$$

$$= \frac{\pi}{\cancel{32}} \times \frac{\cancel{16}\times 16 \times \cancel{4}\, 2}{5}$$

$$\therefore V_2 = \frac{32\pi}{5}$$

$\therefore$ Required Volume

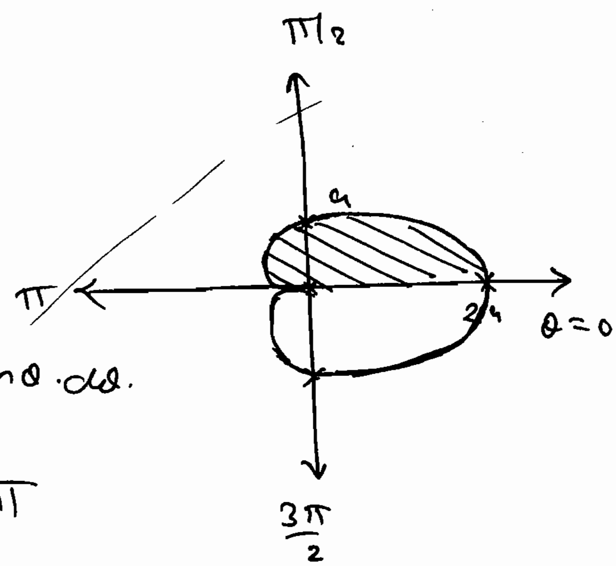
$$V = V_1 - V_2$$

$$= 32\pi - \frac{32\pi}{5}.$$

$$\therefore \boxed{V = \frac{128\pi}{5}}$$

Ex-3 The Volume generated by revolving the Cardioid. $r = a(1+\cos\theta)$ @ the initial line. is ______.

Ans:



$$V = \int_0^{\pi} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$$

$$= \int_0^{\pi} \frac{2\pi}{3} . a^3 (1+\cos\theta)^3 \sin\theta \, d\theta.$$

$$= \frac{2\pi a^3}{3}\left[\frac{-(1+\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= \frac{2\pi a^3}{3} \times 4.$$

$$\therefore \boxed{V = \frac{8\pi a^3}{3}}$$

☆ VECTOR CALCULUS:

⇒ **Scalar function:**

→ If For each value of t, $\phi(t)$ represents a unique scalar then $\phi(t)$ is said to be a Scalar f^n of Scalar variable t.

⇒ **Vector function:**

→ If $\bar{F}(t)$ denotes a unique vector for each value of t, then $\bar{F}(t)$ is said to be a vector f^n of scalar variable t.

⇒ **Position vector:**

→ The Possion vector $P(x, y, z)$ is

$$\boxed{\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}}$$

and $\boxed{r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}}$

In parametric form

$$\boxed{\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}.}$$

* Vector Differentiation

⇒ Derivative of vector function:

→ Any point in the space is represented by $\bar{r}(t)$ then as a value of t varies $\bar{r}(t)$ traces a curve then $\frac{d\bar{r}}{dt}$ at some point on the curve represents a vector along the direction of tangent to the curve

$\bar{F}(t)$

$$\boxed{\frac{d\bar{F}(t)}{dt} = \lim_{\delta t \to 0} \left[\frac{\bar{F}(t+\delta t) - \bar{F}(t)}{\delta t} \right]}$$

NOTE: $\bar{F}(t) \longrightarrow$ const. magnitude.

GTATE

Q: If $\bar{F}(t)$ is a vector with const. mag. the $\boxed{\bar{F}\cdot\frac{d\bar{F}}{dt} = 0}$

$$\bar{F}\cdot\bar{F} = |\bar{F}|^2 = \text{Const.}$$

$$\therefore \frac{d}{dt}(\bar{F}\cdot\bar{F}) = 0$$

$$\therefore \bar{F}\,\frac{d\bar{F}}{dt} + \frac{d\bar{F}}{dt}\cdot\bar{F} = 0$$

$$\therefore \bar{F}\cdot\frac{d\bar{F}}{dt} = 0.$$

NOTE: $\bar{F}(t) \longrightarrow$ Const. direction.

$$\Rightarrow \boxed{\bar{F}\times\frac{d\bar{F}}{dt} = \vec{0}.}$$

⇒ Point Function:

→ If the value of the f^n depends upon the position of the point in the region R of space then it is said to be a point f^n.

* Scalar point function:

→ For each $P(x,y,z)$ in the region R of space if there exist a unique scalar denoted by $\phi(x,y,z)$ then $\phi(x,y,z)$ is a scalar point function. and the Region R so defined is a scalar field.

e.g. The temperature at any pt. on a body.

e.g. for vector pt. f^n.

→ The velocity of a partical in a moving through fluid any time t is a vector pt. function.

⇒ Level Surface:

→ Let, $\phi(x,y,z)$ be a scalar pt. f^n the set of ~~of~~ all points satisfying $\phi(x,y,z)=c$ where, c is an arbitary constant constitute a family of a surfaces called level surfaces.

* Vector

$$\rightarrow \boxed{\nabla = i\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}.}$$

* **Gradiant of a Scalar function:**

$\rightarrow \phi(x,y,z) \rightarrow$ diff. scalar pt. f^n.

$$\boxed{\text{grad } \phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}.}$$

NOTE: If $\phi(x,y,z) = c$ then

$\nabla\phi \rightarrow$ Vector normal to the surface ϕ.

$\frac{\nabla\phi}{|\nabla\phi|} \rightarrow$ unit vector normal to the surface ϕ.

* **Directional derivative (D.D.) :-**

$\rightarrow$ The Directional derivative of a diff^n scalar function in the direction of vector $\bar{a}$ is

$$\boxed{D.D. = \nabla\phi . \frac{\bar{a}}{|\bar{a}|}.}$$

NOTE: Let, $\hat{b} = \frac{\bar{a}}{|\bar{a}|}$.

$$D.D. = \nabla\phi . \hat{b}$$

$$= |\nabla\phi| . |\hat{b}| . \cos\theta.$$

$$= |\nabla\phi| \cos\theta.$$

$\rightarrow$ The max. value of $\cos\theta$ is 1 i.e. when $\theta = 0. \Rightarrow \hat{b}$ should co-inside with $\nabla\phi$ hence

... is max along the

Therefore, $\boxed{\text{Max. value of D.D.} = |\nabla\phi|}$.

(OR)

Greatest rate of increase

* Angle betⁿ the two surfaces:

→ Let, $\phi_1(x,y,z) = c_1$ &

$\phi_2(x,y,z) = c_2$ be two surfaces

and θ be the angle betⁿ them then

$$\boxed{\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|} = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|}}$$

NOTE:

→ The eqⁿ of tangent plane for the surface $\phi(x,y,z) = c$ at a point $P(x_1,y_1,z_1)$ is

$$\boxed{(x-x_1)\frac{\partial\phi}{\partial x} + (y-y_1)\frac{\partial\phi}{\partial y} + (z-z_1)\frac{\partial\phi}{\partial z} = 0}$$

then $\nabla r =$ ______

Ans: $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$.

$\therefore \dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r}$

$\therefore \dfrac{\partial r}{\partial y} = \dfrac{2y}{2\sqrt{x^2+y^2+z^2}} = y/r$

$\dfrac{\partial r}{\partial z} = \dfrac{2z}{2\sqrt{x^2+y^2+z^2}} = z/r.$

$\therefore \nabla r = i\dfrac{\partial r}{\partial x} + j\dfrac{\partial r}{\partial y} + k\dfrac{\partial r}{\partial z}.$

$\therefore \nabla r = i\left(\dfrac{x}{r}\right) + j\left(y/r\right) + k\,(z/r).$

$\therefore \boxed{\nabla r = \dfrac{\bar{r}}{r}}$

Ex-2 $\nabla(r^n)$

$= n r^{n-1} \cdot \dfrac{\bar{r}}{r}.$

$= n \cdot r^{n-2} \cdot \bar{r}.$

NOTE:

$\nabla[f(r)]$

$= f'(r) \cdot \dfrac{\bar{r}}{r}.$

Ex-3 $\nabla(\sin(\log r)) =$ ______

$\rightarrow = \cos(\log r) \cdot \dfrac{1}{r}\,\dfrac{\bar{r}}{r}$

$= \dfrac{\cos(\log r)}{r^2} \cdot \bar{r}$

(4) The unit ...

Ans: $\nabla\phi = \bar{i}(y^3z^2) + \bar{j}(3xy^2z^2) + \bar{k}(xy^3 2z)$

At $(1,-1,2)$.

$\therefore \nabla\phi = -4\bar{i} + 12\bar{j} + -4\bar{k}$.

$\therefore \Rightarrow \bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \cancel{-4i+12j-4k}$

$= \dfrac{-i+3j-k}{\sqrt{1+9+1}}$

$\therefore \boxed{\bar{N} = -\dfrac{i}{\sqrt{11}} + \dfrac{3}{\sqrt{11}}j - \dfrac{1}{\sqrt{11}}k}$

Ex-5 A sphere of unit radius is centred at the origine. A unit vector at a point $P(x,y,z)$ normal to the surface of the sphere is —?

(A) $\left(\dfrac{x}{\sqrt3}, \dfrac{y}{\sqrt3}, \dfrac{z}{\sqrt3}\right)$

(B) $\left(\dfrac{1}{\sqrt3}, \dfrac{1}{\sqrt3}, \dfrac{1}{\sqrt3}\right)$

✓ (c) (x,y,z)

(d) $\left(\dfrac{x}{\sqrt2}, \dfrac{y}{\sqrt2}, \dfrac{z}{\sqrt2}\right)$.

Ans: $\phi = x^2+y^2+z^2-1$.

$\dfrac{\partial\phi}{\partial x} = 2x, \quad \dfrac{\partial\phi}{\partial y} = 2y, \quad \dfrac{\partial\phi}{\partial z} = 2z.$

$\therefore \nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$$\Rightarrow \frac{\nabla\phi}{|\nabla\phi|} = \frac{\quad}{\sqrt{x^2+y^2+z^2}} = \frac{\quad}{\sqrt{1}}$$

$= PEKI'$

NOTE A vector normal to a surface of a sphere at some point with centre at the origin is its position vector.

Ex-6 The Directional derivative of $f = xy^2z$ at $(1,-1,1)$ in the direction of the vector is ____.

$\bar{a} = \bar{i} + \bar{j} + 2\bar{k}$

Ans: $\nabla\phi = y^2z\,i + 2xyz\,\bar{j} + xy^2\,\bar{k}$.

$\therefore$ D.D. $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

$\nabla\phi_{(1,-1,1)} = \bar{i} - 2\bar{j} + \bar{k}$.

$$\therefore \text{D.D.} = \frac{(\bar{i} - 2\bar{j} + \bar{k}) \cdot (i + j - 2k)}{\sqrt{1+1+4}}$$

$$= \frac{1-2-2}{\sqrt{6}}$$

$$\therefore \boxed{\text{D.D.} = -\frac{3}{\sqrt{6}}}$$

Ex-7 The D.D. of $\phi = xy^2 + yz^2 + zx^2$ at $(1,1,1)$ along the direction of tangent to the curve $x = t,\ y = t^2,\ z = t^3$ is ____.

Ans: $\nabla\phi = (y^2 + 2zx)\,i + (2xy + z^2)\,j + (2yz + x^2)\,k$.

$\nabla\phi_{(1,1,1)} = 3\bar{i} + 3\bar{j} + 3\bar{k}$

Vector

$\bar{r}(t) = t\bar{i} + t^2\bar{j} + t^3\bar{k}$.

$\Rightarrow \dfrac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$.

At $(1,1,1)$

$x = t \Rightarrow t = 1$

$t^2 = 1 \Rightarrow t = \pm 1$

$t^3 = 1 \Rightarrow t = 1$.

So, $t = 1$.

$\dfrac{d\bar{r}}{dt} = \bar{i} + 2\bar{j} + 3\bar{k}$.

$\therefore$ D.D. $= \nabla\phi \cdot \dfrac{\bar{r}}{|\bar{r}|}$.

$= \dfrac{(3i + 3j + 3k)(1i + 2j + 3k)}{\sqrt{14}}$.

$= \dfrac{3 + 6 + 9}{\sqrt{14}}$

$$\boxed{D.D. = \frac{18}{\sqrt{14}}}$$

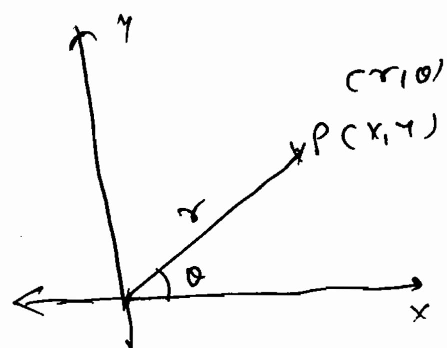
Ex-8 D.D. of $f = \dfrac{y}{x^2+y^2}$ at $(0,1)$ along a direction of a Straight line which makes an angle $\theta = \pi/6$ with positive axis is ___.

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $-\frac{1}{2}$ (D) $-\sqrt{3}/2$.

Ans: $\bar{r} = x\bar{i} + y\bar{j}$.

$\bar{r} = r\cos\theta\bar{i} + r\sin\theta\bar{j}$

$\therefore \dfrac{\bar{r}}{|\bar{r}|} = \cos\theta + \bar{i}\sin\theta$.



$\therefore \quad \hat{e} = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$

$$\nabla f = i\left(\frac{-y^2 x^4}{(x^2+y^2)^2} \right) + j\left[\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} \right].$$

at (0,1).

$\nabla f = -j.$

$\therefore \quad D.D. = \nabla f \cdot \hat{e} = \left(\frac{\sqrt{3}}{2} i + j/2\right)(-j) = -1/2.$

Ex-9 The greatest rate of increase of $\phi = e^{3x} \sin(yz^4)$ at $(0, \frac{\pi}{2}, 1)$ is —?

Ans:
$$\nabla \phi = \bar{i}\left[3e^{3x} \cdot \sin(yz^4)\right] + \bar{j}\left[e^{3x} \cdot \cos(yz^4) \cdot (z^4)\right] + \bar{k}\left[e^{3x} \cdot \cos(yz^4) \cdot 4z^3 y\right].$$

At $(0, \frac{\pi}{2}, 1)$.

$\nabla \phi = \bar{i}[3] + \bar{j}[0]\bar{j} + [0]\bar{k}$

$\nabla \phi = 3\bar{i}$

$\therefore$ max. value of $D.D. = |\nabla \phi| = 3.$

Ex-10 The D.D. of $f = ax^2 + by^2 + cz^2$ at $(1,1,2)$ has max. magnitude 4 along the direction ∥ee to y-axis then the values of a, b, c —?

(A) 0, 1, 0 (B) 0, 2, 0 (C) 0, 4, 0 (d) 1, 0, 1.

Ans: $\nabla f = 2ax\bar{i} + 2by\bar{j} + 2cz\bar{k}.$

at $(1,1,2)$, $\nabla f = 2ai + 2bj + 4c\bar{k}.$

$\Rightarrow \quad a=0, \quad c=0.$

$\Rightarrow \quad \nabla f = 2bj.$

$|\nabla f| = 4.$

$\therefore \quad 2b = 4, \Rightarrow b = 2$

* Divergence of a Vector function:

$\rightarrow \quad \bar{F}(x,y,z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$\downarrow$

diff. Vector f^n.

$$\boxed{\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.}$$

NOTE: If $\boxed{\nabla\cdot\bar{F}=0}$ then $\bar{F}$ is said to be solenoidal vector.

* Curl of a Vector function:

$$\boxed{\therefore \operatorname{curl} F = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

Let, $\bar{v} \rightarrow$ linear velocity

$\bar{\omega} \rightarrow$ angular velocity

$$\boxed{\bar{V} = \bar{\omega}\times\bar{r}}$$

NOTE: $\nabla\times(\bar{a}\times\bar{r}) = 2\bar{a}$

$\operatorname{curl}\bar{v} = \nabla\times(\bar{\omega}\times\bar{r})$

$= 2\bar{\omega}$

$\bar{\omega} = \frac{1}{2}\operatorname{curl}\bar{v}$

NOTE: If $\boxed{\text{curl } F = 0}$ then F is said to be irrotational vector.

* Scalar Potential function:

→ If F is irrotational then ∃ a scalar function $\phi(x,y,z)$ such that $\boxed{\bar{F} = \nabla \phi}$, then ϕ is said to be Scalar potential function.

NOTE:

$$\boxed{\phi(x,y,z) = \int_a^x F_1(x,y,z)\,dx + \int_b^y F_2(a,y,z)\,dy + \int_c^z F_3(a,b,z)\,dz.}$$

NOTE:

① curl (grad ϕ) = $\bar{0}$.

② Div (curl $\bar{F}$) = 0.

③ Div (grad ϕ) = $\nabla \cdot (\nabla \phi)$.

$= \nabla^2 \phi$

(*) Laplacian operator $= \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$.

④ Curl (curl $\bar{F}$) = $\nabla \times (\nabla \times \bar{F})$

$= \nabla(\nabla \cdot \bar{F}) - (\nabla \cdot \nabla)\bar{F}$

$=$ grad (div $\bar{F}$) $- \nabla^2 \bar{F}$.

⑤ div $(\bar{A} \times \bar{B}) = \bar{B} \cdot \text{curl } \bar{A} - \bar{A} \cdot \text{curl } \bar{B}$

Ex-1 If $\bar{F} = (4x^2 - yz)i - y^2 z j + xz k$ represent a velocity vector then

① $div \bar{F}$ at $(3,-1,2)$ is ___.

② Its corresponding angular velocity at $(1,-1,1)$ is ___.

Ans: ⑧

(i) $div \bar{F} = (8x \cancel{+2}) \cancel{i}$

$= 8x i - 2yz j + x k$

at $(3,-1,2)$

$div \bar{F} = 24 i + 4 j + 3$

$\therefore \boxed{div \bar{F} = 31}$

(ii) $\bar{\omega} = \frac{1}{2} curl \bar{F}$

$$curl \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x^2 - yz & -y^2 z & xz \end{vmatrix}$$

$= i[0 + y^2) - j[z + y] + k[0 + z].$

$curl \bar{F} = y^2 i - (y+z) j + z k$

at $(1,-1,1)$

$curl \bar{F} = 1 i - 0 j + k = i + \bar{k}.$

$\therefore \bar{\omega} = \frac{1}{2} curl \bar{F}$

$\boxed{\bar{\omega} = \frac{1}{2}(\bar{i} + \bar{k})}$

Vector f^n $\bar{F} = (\lambda x^2 y - yz)\bar{i} + (xy^2 - xz^2)\bar{j} + (2xyz + x^2y^2)\bar{k}$ is ~~Solenoi~~ Solenoidal is ______.

Ans:

For Solenoidal

$\nabla \cdot \bar{F} = 0.$

$\therefore \quad 2\lambda xy + 2xy + 2xy + 2xy^2 = 0.$

$2xy \quad (\lambda + 1 + 1) = 0$

$\boxed{\lambda = -2}$

Ex-3 If $\bar{F} = (x + 2y - az)\bar{i} + (bx - y + 4z)\bar{j} + (3x + cy - z)\bar{k}$ is irrotational.

then Values of a, b, c are ______.

Ans: Curl $\bar{F} = 0$ for irrotational.

$$\therefore \text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y-az & bx-y+4z & 3x+cy-z \end{vmatrix} = 0.$$

$\therefore \; i(c-4) - j(3+a) + k(b-2) = 0.$

So, $c = 4,$

$a = -3.$

$b = 2.$

Ex-4 If $\phi = xyz$ then ...

ⓐ Solenoidal ⓒ a & b

ⓑ irrotational ⓓ none.

Ans: $\nabla \phi = yz\,i + xy\,j + xy\,\bar{k}$.

$\therefore$ div $(\nabla \phi) = 0 + 0 + 0 = 0$.

so, solenoidal.

curl $(\nabla \phi) =$ curl (grad ϕ) $= \bar{0}$.

Ex-5 If $\bar{F} = 4x\bar{i} - (y^2 + 2z)\bar{j} + z^2 . \bar{k}$ then the value of $\nabla . (\nabla \times \bar{F})$ at $(3, -1, 4)$ is ____.

Ans: div (curl $\bar{F}$) $= 0$.

Ex-6 If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, and $r = |\bar{r}|$ then for what value of n the vector fn $r^n . \bar{r}$ is solenoidal.

(a) $n = 3$ (b) $n = 2$

(c) $n = -3$ (d) $n = -2$.

Ans: $r^n . \bar{r} = \underbrace{r^n x \bar{i}}_{F_1} + \underbrace{r^n y \bar{j}}_{F_2} + \underbrace{r^n z \bar{k}}_{F_3}$.

$\rightarrow$ div $(r^n . \bar{r}) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

$\therefore \dfrac{\partial F_1}{\partial x} = r^n (1) + x . n . r^{n-1} . \dfrac{\partial r}{\partial x}$.

$r = \sqrt{x^2 + y^2 + z^2}$

$\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}}$

$= x / r$

$= r^n + x.n.r \cdot \left(\frac{\hat{x}}{r}\right)$

$\frac{\partial r}{\partial x} = r^n + x^2 . n r^{n-2}.$

$\therefore \frac{\partial F_2}{\partial y} = r^n + n r^{n-2} . y^2.$

$\frac{\partial F_3}{\partial z} = r^n + n . r^{n-2} . z^2.$

$\text{div}(r^n . \bar{r}) = 3r^n + n r^{n-2} (x^2+y^2+z^2)$

$= 3r^n + n r^n.$

$$\boxed{\text{div}(r^n . \bar{r}) = (n+3)\, r^n}$$

$\text{div}(r^n . \bar{r}) = 0 \Rightarrow \boxed{n = -3}$

✯ VECTOR ✯ Integration:

* Line Integral:

→ If $\bar{F}(x,y,z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is a diffⁿ vector fⁿ defined ~~the~~ along the curve C then its line integral along C is

$$\boxed{\int_C \bar{F} \cdot d\bar{r}}$$

⇒ In cartesian form.

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz}.$$

→ If $\underline{c}$ is a closed curve then line integral of $\underline{\bar{F}}$ along c is called circulation of $\bar{F}$.

denoted by,
$$\boxed{\oint_c \bar{F}\cdot d\bar{r}}$$

⇒ Work done by Force:

→ The total work done by Force in moving a partical along c is

$$\therefore \boxed{W.D. = \int_c \bar{F}\cdot d\bar{r}}$$

NOTE:

→ The Line integral of an irrotational vector f^n is Independent of the path of the curve.

→ If $\bar{F}$ is irrotational

$\boxed{\bar{F} = \nabla\phi}$ where, ϕ is a scalar potential f^n.

then
$$\boxed{\int_A^B \bar{F}\cdot d\bar{r} = \phi_B - \phi_A}$$

Ex-1 The value of $\int_C \bar{F}\cdot d\bar{r}$, $\bar{F} = x^2 y\,\bar{i} - xy^2\,\bar{j}$.

Ans: $C \to y = 2x^2$ joining the point $(0,0) \to (1,2)$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C x^2 y\,dx - x^2 y^2\,dy.$$

$$\rightarrow \quad y = 2x^2$$
$$dy = 4x\,dx$$

$$\int \bar{F}\cdot d\bar{r} = \int_0^1 2x^4\,dx - x^2(2x^2)(4x)\,dx$$

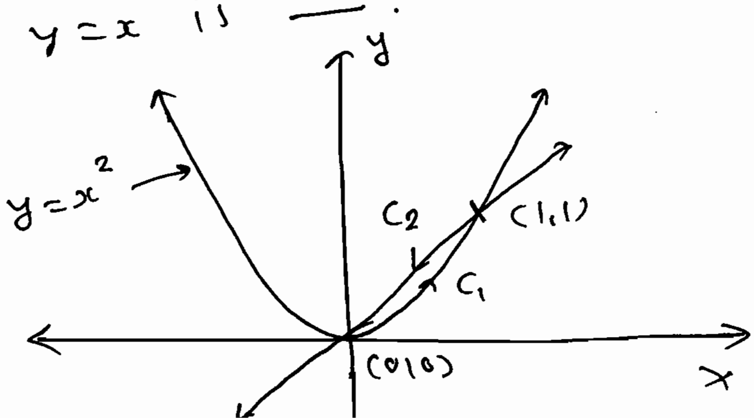
$$= \int_0^1 \left(2x^4 - 8x^5\right) dx$$

$$= \left[\frac{2x^5}{5} - \frac{8x^6}{6}\right]_0^1$$

$$= \frac{2}{5} - \frac{16}{7} = \frac{14-80}{35} = \frac{-66}{35}.$$

$$\cancel{\frac{12-40}{30}}$$
$$\cancel{= \frac{-28}{30}}$$
$$\cancel{= -14}$$

Ex-2 $\int_c \bar{F}\cdot d\bar{r}$, $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ and $C \to y = x^2$ and $y = x$ is ___.



$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}_1 \cdot \quad \int_{C_2} \bar{F}_2$$

(i) Along C_1:

$y = x^2 \Rightarrow dy = 2x\,dx.$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int 3xy\,dx - y^2 dy$$

$$= \int_0^1 3x(x^2)\,dx - x^4 . 2x\,dx.$$

$$= \left[3\frac{x^4}{4} - 2\frac{x^6}{6}\right]_0^1$$

$$= 3/4 - \frac{1}{3}$$

$$= \frac{5}{12}.$$

(ii) Along C_2:

$y = x \Rightarrow dy = dx.$

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_1^0 3x(x)\,dx - x^2 dx.$$

$$= \left[x^3 - x^3/3\right]_1^0$$

$$= \overset{-1}{0} + 1/3.$$

$$= -2/3.$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \frac{5}{12} - 2/3 = -3/12. = -\frac{1}{4}.$$

$\boxed{\text{NOTE:}}$ → The line integral of an irrotational vector f^n over any closed curve is zero.

where c is the circle $x^2+y^2=4$ in xy plane is ____.

Ans: Let, $x = 2\cos t$, $y = 2\sin t$.

$\Rightarrow dx = -2\sin t\, dt$ $dy = 2\cos t\, dt$

$$I = \int_0^{2\pi} (2\cos t + 4\sin t)(-2\sin t\, dt) - (4\cos t - 6\sin t)(2\cos t)\, dt.$$

$$I = \int_0^{2\pi} (8\sin t\cdot\cos t - 8\sin^2 t \mp 8\cos^2 t)\, dt.$$

$$I = \int_0^{2\pi} (4\sin 2t - 8)\, dt.$$

$$I = \left[-4\frac{\cos 2t}{2} - 8t\right]_0^{2\pi}$$

$\therefore \boxed{I = -16\pi.}$

Ex-4 The total W.D. by the Force $\bar{F} = (3x^2+6y)\bar{i} - (14yz\bar{j}) + 20xz^2\bar{k}$. in moving a particle along a straight line joining the set off point $(0,0,0)$ and $(1,1,2)$ is ____.

Ans: $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

$$\therefore \frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{2-0}.$$

$\therefore x = t$, $y = t$, $z = 2t$.

$\therefore dx = dt$, $dy = dt$, $dz = 2dt$.

$$W.D. = \int_c \bar{F}\cdot d\bar{r}$$

$$= \int_c (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz.$$

$$= \int_0^1 (3t^2+6t)\,dt - 28t^2\,dt + 160t^3\,dt.$$

$$W.D. = -\frac{25}{2} + \frac{6}{2} + \frac{160}{4} = \cancel{\ } - \frac{19}{2} + \frac{160}{4}$$

$$= \frac{-38+160}{4}$$

$$\therefore \boxed{W.D. = \frac{122}{4}}.$$

Ex-5 The value of the line integral $\int \bar{V}\cdot d\bar{r}$ of the vector fn $\bar{V} = yz\hat{i} + (xz+1)\hat{j} + xy\hat{k}$. from $(0,0,1)$ to $(2,1,4)$ is ____.

(A) 7 (C) 9

(B) 8 (D) cann't be determine without specifying the path.

Ans: 5

$$\text{Curl } \bar{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ yz & xz+1 & xy \end{vmatrix}$$

$$= i[x-x] - j[\cancel{y}-y] + k[z-z]$$

$$= \bar{0}$$

$\Rightarrow \bar{V}$ is irrotational.

$$\Rightarrow \quad \bar{V} = \nabla \phi .$$

$$\phi = \int_a^x F_1(x,y,z)\,dx + \int_b^y F_2(a,y,z)\,dy + \int_c^z F_3(a,b,z)\,dz$$

$$= \int_a^x yz\,dx + \int_b^y (az+1)\,dy + \int_c^z ab\,dz .$$

$$= [yzx]_a^x + [azy+y]_b^y + [abz]_c^z .$$

$$= xyz - \cancel{ayz} + \cancel{ayz} + y - \cancel{abz} - b + \cancel{abz} - abc .$$

$$\phi = xyz + y + k \quad , \quad k = -abc - b$$

$$\therefore \int \bar{V}\cdot d\bar{r} = \phi(2,1,4) - \phi(0,0,1)$$

$$= 8 + 1 + k - (0 + 0 + k).$$

$$=$$

$$\therefore \boxed{\int \bar{V}\cdot d\bar{r} = 9}$$

☆ Green Theorem for a Plane:

→ Let, $M(x,y)$ and $N(x,y)$ be the continous function having contineous first order partial derivatives defined in the closed region R bounded by the closed curve C then

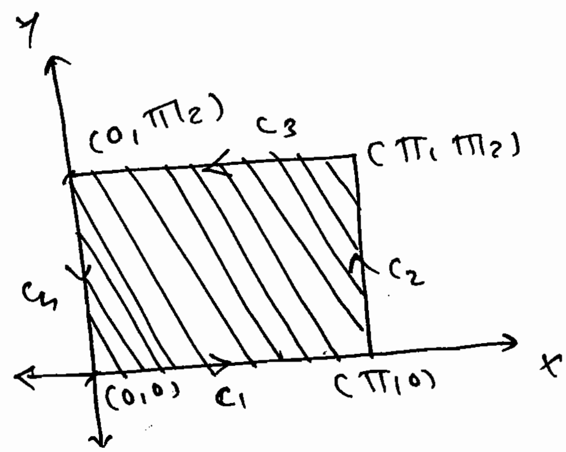
$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy.}$$

✓

Ex-1 The value of $\int_C e^{-x}\cos y\,dx - e^{-x}\cdot\sin y\,dy$ where C is the Rectangle with the vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.

Ans: $M = e^{-x}\cdot\cos y$

$N = -e^{-x}.\sin y.$



$$\frac{\partial M}{\partial y} = -e^{-x}\cdot\sin y.$$

$$\frac{\partial N}{\partial x} = e^{-x}\cdot\sin y.$$

$$I = \int_0^{\pi/2}\int_0^{\pi} 2\,e^{-x}\sin y\,dx\,dy.$$

$$I = \int^{\pi/2} \sin y \left[\frac{e^{-x}}{-1}\right]_0^{\pi} dy$$

$$I = 2\int_{0}^{\pi/2} \sin y . \left[1 - e^{-\pi}\right]$$

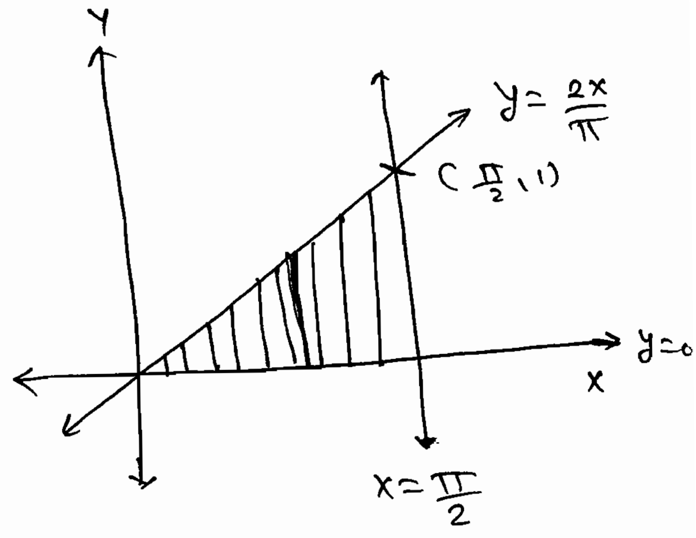
$$= 2\left(1 - e^{-\pi}\right) \left(-\cos y\right)_{0}^{\pi/2}$$

$$\therefore \boxed{I = 2\left(1 - e^{-\pi}\right)}$$

Ex-2 The value of $\int_C (y - \sin x)dx + (\cos ...$

C is a curve bounded by $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2x}{\pi}$.

is ____.



Ans:
$y=0$ to $y=\frac{2x}{\pi}$

$x=0$ to $x=1$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x.$$

$$\therefore -\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = -(\sin x + 1).$$

$$\therefore I = \int_0^1 \int_0^{2x/\pi} -(1+\sin x) \cdot dy\, dx$$

$$= \int_0^1 -(1+\sin x)\frac{2x}{\pi}\, dx$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - (1)(-\sin x)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8} + 1\right]$$

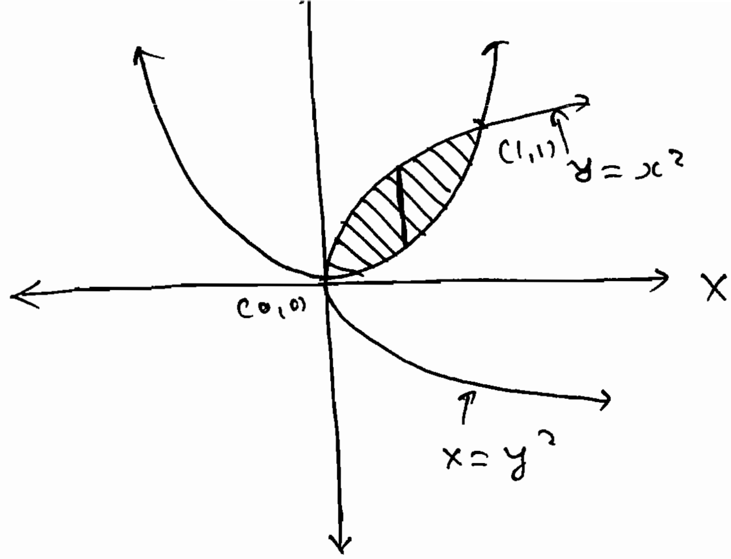
Ex-3 $\oint_C (4xy - 3y^2)\,dy + (3x^2 - 8y^2)\,dx$

$C \to y = x^2$ and $y = \sqrt{x}$.

Ans: $\frac{\partial M}{\partial y} = \cancel{4x} -16y.,\quad \frac{\partial N}{\partial x} = 8x .+4y.$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = +20y.$$

$$I = \int_{0}^{1} \int_{x^2}^{\sqrt{x}} +20y \cdot dy\, dx$$



$$I = \int_{0}^{1} \left[+10y^2\right]_{x^2}^{\sqrt{x}}$$

$$= 10 \int_{0}^{1} \left[x - x^4\right] dx$$

$$= 10 \left[\frac{1}{2} - \frac{1}{5} \right].$$

$$= 10 \left[\frac{3}{10} \right]$$

$$\therefore \boxed{I = 3}$$

* Surface Integral:

→ Let, $\bar{F}(x, y, z) = F_1 i + F_2 j + F_3 k$ define over a surface S, then its surface integral is

$$\int_S \bar{F} \cdot d\bar{s} = \int \bar{F} \cdot \bar{N} \cdot ds$$

where, $\bar{N}$ is unit outword drawn normal to the surface S.

→ In Cartesian form,

$$\int_S \bar{F} \cdot \bar{N} \cdot ds = \int_S F_1\, dy\, dz + \int_S F_2\, dx\, dz + \int_S F_3\, dx\, dy.$$

→ ① If R_1 is the Projection of 's' onto xy plane then

$$\boxed{\int_S \bar F\cdot\bar N\,ds = \iint_{R_1} \bar F\cdot\bar N\,\frac{dx\,dy}{|\bar N\cdot\bar k|}}$$

② Similarly $R_2 \to$ yz-plane.

$$\boxed{\int_S \bar F\cdot\bar N\,ds = \iint_{R_2} \bar F\cdot\bar N\,\frac{dy\,dz}{|\bar N\cdot\bar i|}}$$

③ $R_3 \to$ xz-plane.

$$\boxed{\int_S \bar F\cdot\bar N\,ds = \iint_{R_3} \bar F\cdot\bar N\cdot\frac{dx\,dz}{|\bar N\cdot\bar j|}}$$

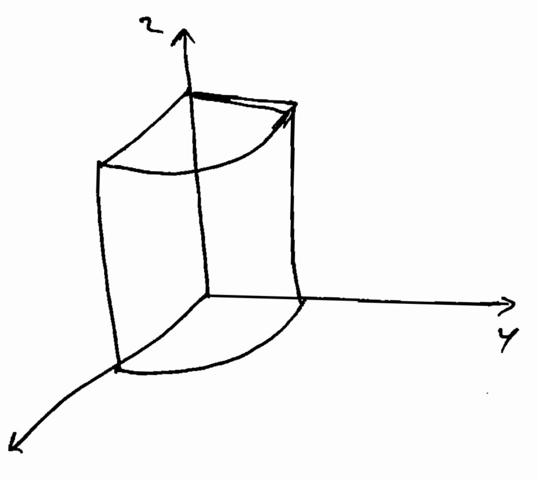
Ex-1 The value of $\int_S \bar F\cdot\bar N\,ds$ where $\bar F = z\bar i + x\bar j - 3y^2z\bar k$ and s is the surface of the cylinder $x^2+y^2=16$ included into the first octant betn $z=0$ & $z=5$. is ___.

Ans: Let, $\phi = x^2+y^2$

$\nabla\phi = 2xi + 2yj$

$\therefore \bar N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{xi+yj}{\sqrt{x^2+y^2}}$

$= \frac{x}{4}i + \frac{y}{4}j$



$\bar{F}\cdot\bar{N} = \frac{xz}{4} + \frac{xy}{4}.$

$\bar{F}\cdot\bar{N} = \frac{x}{4}(y+z).$

Let, $R \to yz$ - plane.

$$\int_S \bar{F}\cdot\bar{N}\,ds = \iint_R \bar{F}\cdot\bar{N}\,\frac{dy\,dz}{|\bar{N}\cdot\bar{i}|}.$$

$$= \int_{z=0}^{5}\int_{y=0}^{4} \cancel{\frac{x}{4}}\,(y+z).\,\frac{dy\,dz}{\cancel{x/4}}$$

$$= \int_0^5 \left[\frac{y^2}{2} + yz\right]_0^4.$$

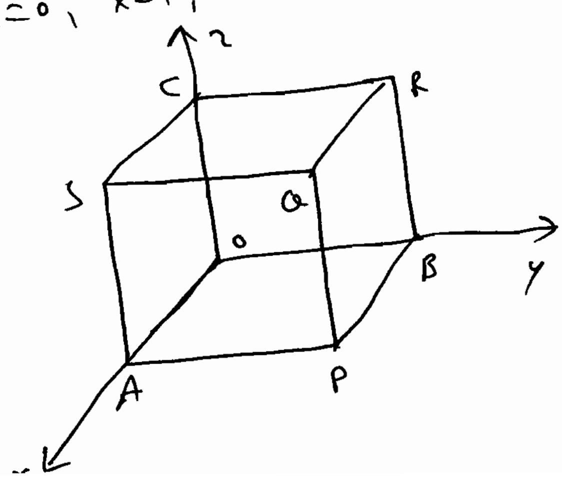
$$= \int_0^5 (8 + 4z)\,dz$$

$$= \left[8z + 2z^2\right]_0^5$$

$$= 40 + 50$$

$$\therefore \boxed{I = 90}$$

Ex-3 $\int_S \bar{F}\cdot\bar{N}\,ds$, $\bar{F} = 4xzi - y^2j + yzk$, S is surface bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$



Ans: $\int_S \bar{F}\cdot\bar{N}\,ds$

$$= \int_{S_1} + \int_{S_2} + \cdots \int_{S_6}$$

$\therefore \quad \bar{N} = -\bar{k}, \quad , z=0, \quad \bar{F}\cdot\bar{N} = -yz = 0.$

$$\int_{S_1} \bar{F}\cdot\bar{N}\, ds = 0.$$

(ii) over CSQR ($\parallel^{el}$ to x-y- plane).

$\therefore \quad \bar{N} = \bar{k}.$

$z = 1, \quad \bar{N} = \bar{k}, \quad \bar{F}\cdot\bar{N} = yz = y.$

$$\int_{S_2} \bar{F}\cdot\bar{N}\, ds = \int_{S_2} y\, ds = \iint_R y \frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}.$$

$$= \int_0^1\int_0^1 y\, dy\, dx.$$

$$= \frac{1}{2}\times 1 = \frac{1}{2}.$$

(iii) over S_3: In y-z plane.

$\therefore \quad \bar{N} = -\bar{i}, \quad x = 0.$

$\therefore \quad \bar{F}\cdot\bar{N} = -4xz = 0.$

$$\int_{S_3} \bar{F}\cdot N\bar{s} = 0.$$

(iv) over S_4: $\parallel^{el}$ to yz plane. (SAPQ).

$\bar{N} = \bar{i}, \quad x = 1$

$\bar{F}\cdot\bar{N} = 4xz = 4z.$

$$\int_S \bar{F}\cdot\bar{N} = \int_0^1\int_0^1 4z.\, dy\, dz = 4\times\frac{1}{2}\times 1$$

$$= 2.$$

(V) over S_5. In xz-plane

$\rightarrow$ $\bar{N} = -\bar{j}$, $y = 0$.

$\bar{F} \cdot \bar{N} = y^2 = 0$.

$$\int_{S_5} \bar{F} \cdot \bar{N} \, ds = 0.$$

(VI) $\parallel^{el}$ to xz- plane (OR BP).

$\rightarrow$ $\bar{N} = \bar{j}$, $y = 1$.

$\bar{F} \cdot \bar{N} = y^2 = 1$.

$$\int_{S_6} \bar{F} \cdot \bar{N} \, ds = \int_0^1 \int_0^1 - - \cancel{x} \, dx \, dz$$

$$= -1.$$

$$\therefore \int_S \bar{F} \cdot \bar{N} \, ds = \int_{S_1} + \int_{S_2} + \cdots \int_{S_6}$$

$$= 0 + \frac{1}{2} + 0 + 2 + 0 - 1$$

$$= 3/2.$$

(OR)

$$\rightarrow \int_S \bar{F} \cdot \bar{N} \, ds = \int_V \operatorname{div} \bar{F} \cdot dV$$

$$\operatorname{div} \bar{F} = 4z - 2y + y = 4z - y$$

$$\int_S \bar{F} \cdot \bar{N} \, ds = \int_V (4z - y) \, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y) \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^1 (2-y)\, dy\, dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 2 - \frac{1}{2}\, dx$$

$$\therefore \boxed{I = 3/2.}$$

☆ Volume Integral:

→ Let $\phi(x,y,z)$ be a diffⁿ scalar function and $\bar{F}(x,y,z)$ be a diffⁿ vector function defined over a region whoes volume bounded is P then the Volume Integrals are

$$\int_V \phi\, dv \quad \text{and} \quad \int_V \bar{F} \cdot dv$$

$$= \bar{i} \int_V F_1\, dv + \bar{j} \int_V F_2\, dv + \bar{k} \int_V F_3\, dv$$

Ex-1 The vame of $\int_V (2x+y)\, dv$ where, v is the region bounded by $x=0$, $y=0$, $y=z$, $z=x^2$ and $z=4$ is ____.

Ans: $z = x^2$, $z=4$

$\therefore x^2 = 4$

$\boxed{x = 2}$

$$I = \int_0^2 \int_0^2 \int_{x^2}^4 (2x+y)\, dz\, dy\, dx.$$

$$= \int_0^2 \int_0^2 \left[2xz + yz\right]_{x^2}^{4} \cdot dy\, dx$$

$$= \int_0^2 \int_0^2 (8x + \cancel{8}4y - 2x^3 - yx^2)\, dy \cdot dx$$

$$= \int_0^2 \left[8xy + 2y^2 - 2yx^3 - \frac{x^2y^2}{2}\right]_0^2 dx$$

$$= \int_0^2 16x + 8 - 4x^3 - 2x^2 \quad dx$$

$$= \left[8x^2 + 8x - x^4 - \frac{2}{3}x^3\right]_0^2$$

$$= 32 + \cancel{16} - \cancel{16} - \frac{16}{3}$$

$$= 16\,[\,5/3)$$

$$\therefore \boxed{I = 80/3.}$$

☆ Gauss - Divergence Theorem

→ Let, S be a closed surface enclosing a Volume V and $\bar{F}(x,y,z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a diffⁿ vector fⁿ defined over the Surface S then

$$\boxed{\int_S \bar{F}\cdot\bar{N}\,ds = \int_V \operatorname{div}\bar{F}\,dV.}$$

→ In cartesian form,

$$\boxed{\int_S F_1\,dydz + F_2\,dxdz + F_3\,dxdy = \int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV.}$$

Ex-1 If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then the value of $\int_S \bar{r}\cdot\bar{N}\,ds$, where s is a closed surface enclosing a Volume V is ——.

(A) V (B) 2V (C) 3V ✓ (D) 4V

Ans: $\operatorname{div}\bar{r} = 1 + 1 + 1 = 3.$

$$\int_S \bar{r}\cdot\bar{N}\,ds = \int_V \operatorname{div}\bar{r}\,dv$$

$$= \int_V 3\,dv = 3V.$$

Ex-2 $\int x\,dy\,dz + y\,dx\,dz + z\,dx\,dy$ where ...

Surface of (1) cylinder bounded by $y^2+z^2=9$, $x=0$ and $x=2$.

(2) Sphere $x^2+y^2+z^2=4$.

(3) S bounded by $x=0$, $z=0$, $x+y+z=1$.

Ans: $\int_S x\,dy\,dz + \int y\,dx\,dz + \int z\,dx\,dy = \int_S \bar{r}\cdot\bar{N}\,ds$

$= 3V.$

(i) $3V = 3\times\pi r^2 h = 3\times\pi\times(3)^2\cdot(2) = 54\pi$

(ii) $3V = 3\times\frac{4}{3}\pi r^3 = 4\times\pi\times(2)^3 = 32\pi.$

(iii) $3V = 3\int_0^1\int_0^{1-x}\int_0^{1-x-y} 1\;dz\,dy\,dx$

$$= 3\int_0^1\int_0^{1-x}(1-x-y)\cdot dy\,dx$$

$$= 3\int_0^1\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}\cdot dx$$

$$= 3\int_0^1\left((1-x)-x(1-x)-\frac{(1-x)^2}{2}\right)dx.$$

$$= 3\int_0^1\frac{(1-x)^2}{2}\cdot dx$$

$$= 3 \int_0^1 \frac{(1-x)^2}{2} \cdot dx$$

$$= 3 \times \left[-\frac{(1-x)^3}{6} \right]_0^1$$

$$= 3 \times \left[0 + \frac{1}{6} \right].$$

$$\therefore \boxed{I = \frac{1}{2}.}$$

Ex-3 The value of $\int_S (x^2 + 2y^2 + 3z^2)\, ds$ where S is a surface of a unit sphere with centre at the origin is ___.

Ans: Here, $\bar{F} \cdot \bar{N} = x^2 + 2y^2 + 3z^2.$

Now, we nee $\bar{N}$ so

Let, $\phi = x^2 + y^2 + z^2 - 1.$

$$\therefore \nabla \phi = 2x i + 2y j + 2z k$$

$$\bar{N} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x i + 2y j + 2z k}{2(\sqrt{x^2 + y^2 + z^2})}$$

$$\bar{N} = x i + y j + z k$$

$$\therefore \bar{F} \cdot \bar{N} = F_1 x + F_2 y + F_3 z = x^2 + 2y^2 + 3z^2.$$

$$\therefore F_1 = x, \quad F_2 = 2y, \quad F_3 = 3z.$$

$$\therefore \bar{F} = x\bar{i} + 2y\bar{j} + 3z\bar{k}$$

$$\therefore \operatorname{div} \bar{F} = 1 + 2 + 3 = 6.$$

$$\int_S (x^2 + 2y^2 + 3z^2)\, ds = \int_V \operatorname{div} \bar{F} \cdot dv = \int_V 6 = 6v$$

$$= 6 \times \frac{4}{3} \pi r^3 = 6 \times \frac{4}{3} \pi (1)^3$$

Ex-4 $\int_S \text{curl } F \cdot \bar{N}\, ds$, $F =$ ...

$x=0,\ y=0,\ z=0\ ,\ x+y+z=1.$

Ans: $I = \int_S \text{curl } \bar{F}\cdot\bar{N}\, ds = \int_V \text{div}\,(\text{curl } \bar{F})\, dv$

$= 0.$

$(\because \text{div}\,(\text{curl } \bar{F}) = 0).$

Ex-5 $\int_S \bar{F}\cdot\bar{N}\, ds$, where $\bar{F} = 4x^2\bar{i} - yz^2\bar{j} + xz\bar{k}$.

and s is a surface bounded by $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$ is ——.

Ans: $\text{div}\,\bar{F} = 8x - z^2 + x.$

$$\int_S \bar{F}\cdot\bar{N}\, ds = \int_V \text{div}(\bar{F})\, dv$$

$$= \int_0^1\int_0^2\int_0^3 (8x - z^2 + x)\, dz\, dy\, dx.$$

$$= \int_0^1\int_0^2 \left[9xz - \frac{z^3}{3}\right]_0^3 dy\, dx.$$

$$= \int_0^1\int_0^2 27x - 9\, .\, dy\, dx.$$

$$= \int_0^1 \left[27xy - 9y\right]_0^2 dx.$$

$$= \int_0^1 54x - 18 \cdot dx$$

$$= \left[27x^2 - 18x\right]_0^1$$

$$= 27 - 18$$

$$\therefore \boxed{I = 9}$$

Ex-6 The value of $\int_S \bar{F}\cdot\bar{N}\, ds$, where $\bar{F} = 4xz\bar{i} - 2y^2\bar{j} + z^2\bar{k}$.

and S is the surface bounded by $x^2+y^2=4$, $z=0$ & $z=3$ is ______.

(A) 64π (B) 84π (C) 104π (D) None.

Ans:

$$\operatorname{div}\bar{F} = 4 - 4y + 2z.$$

$$\therefore \int_S \bar{F}\cdot\bar{N}\cdot ds = \int_V \operatorname{div}(\bar{F})\cdot dV = \int_V (4-4y+2z)\, dV$$

$$I = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\, dz\, dy\, dx$$

$$\therefore I = \int_0^3\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-4y+2z)\, dy\, dx\, dz.$$

let, $x = r\cos\theta$, $y = \sin\theta$, $|J| = r$, $z=z$

$$I = \int_0^3\int_0^{2\pi}\int_{r=0}^{2} (4 - 4r\sin\theta + 2z)\, r\, dr\, d\theta\, dz.$$

$$= \int_0^3 \int_0^{2\pi} \left[(4+2z)\frac{r^2}{2} - 4\sin\theta \frac{r^3}{3}\right]_0 d\theta\, dz.$$

$$= \int_0^3 \int_0^{2\pi} (8+4z) - \frac{32\sin\theta}{3} \, d\theta\, dz.$$

$$= \int_0^3 \left[(8+4z)\theta + \frac{32\cos\theta}{3}\right]_0^{2\pi} dz$$

$$= \int_0^3 \left[(8+4z)2\pi + \frac{3\cancel{2}}{\cancel{3}} - \frac{\cancel{32}}{3}\right] dz$$

$$= 2\pi\left[8z + 2z^2\right]_0^3$$

$$= 2\pi\left[24 + 18\right].$$

$$\therefore \boxed{I = 84\pi}$$

STOKE'S THEOREM

[Line Integral $\Longleftrightarrow$ surface integral].

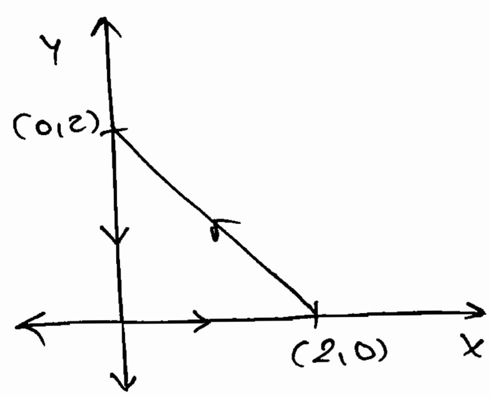
→ Let, S be an open surface bounded by a closed curve C and $\bar{F}(x, y, z)$ be a diffn vector fn define along a curve C then

$$\boxed{\oint_C \bar{F} \cdot d\bar{r} = \int_C \text{curl}\, \bar{F} \cdot \bar{N}\, ds}$$

i.e. $$\boxed{\oint_C F_1\,dx + F_2\,dy + F_3\,dz = \int_S (\nabla \times \bar{F}) \cdot \bar{N}\, ds.}$$

Ex-1 $\oint_C \bar{F} \cdot d\bar{r}$, $\bar{F} = yz\,\bar{i} + xz\,\bar{j} + xy\,\bar{k}$ where, C is a curve bounded by $x=0, y=0, x+y=2$ in xy-plane is ____.

Ans: curl F



$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix}$$

$$= i[x - x] - j[y - y] + k[z - z]$$

$$= \bar{0}$$

$\Rightarrow \bar{F}$ is irrotational.

$$\oint \bar{F} \cdot d\bar{r} = \int_C \text{curl}\,(\bar{F} \cdot \bar{N})\, ds = \int_S \bar{0} \cdot \bar{N}\, ds = 0.$$

Ex-2 The value $\oint_c$ ... and c is the boundary of the circular disk $x^2+y^2 \le 1$, $z=0$

Ans: $$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & z \end{vmatrix}$$

$$= i[0-0] - j[0+\cancel{3y^2}] + k[3x^2+3y^2]$$

$$= 3(x^2+y^2)\bar{k}.$$

$\bar{N}=\bar{k}$, $\text{curl}\cdot\bar{F}\cdot\bar{N} = 3(x^2+y^2)$

$\cancel{\oint_c \text{curl}\,\bar{F}/\bar{N}}$ $$\oint \bar{F}\cdot d\bar{r} = \int_c \text{curl}\;\bar{F}\cdot\bar{N}\,ds.$$

$$= \int_s 3(x^2+y^2)\,ds.$$

$$= \iint_R 3(x^2+y^2)\frac{dx\cdot dy}{|\bar{N}\cdot\bar{k}|}.$$

Let, $x = r\sin\theta$, $y = r\cos\theta$
$\Rightarrow x^2+y^2 = r^2$, $|J| = r$.

$$I = \int_{\theta=0}^{2\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta.$$

$$= \frac{1}{4}\times 3\times 2\pi$$

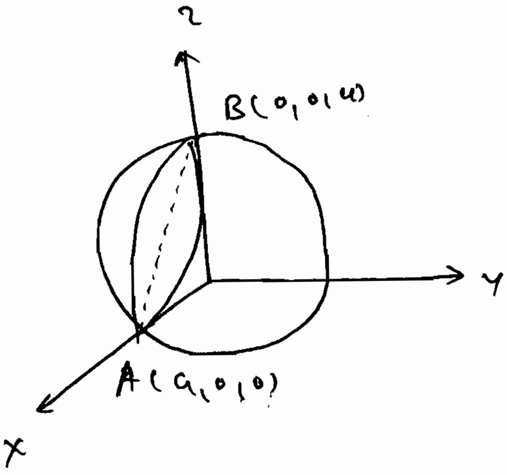
$$\therefore \boxed{I = \frac{3\pi}{2}}$$

Ex-3

$\oint_C$ ...

of intersection of $x^2+y^2+z^2 = a^2$, & $x+z=a$

is ——.



$\rightarrow$ The intersection the sphere $x^2+y^2+z^2=a^2$ with the plane $x+z=a$ is a circle in the plane $x+z=a$ with $\overline{AB}$ as diameter.

$$AB = \sqrt{2a^2} = \sqrt{2}a.$$

$$\therefore r = \frac{a}{\sqrt{2}}.$$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= i(0-1) - j(1-0) + k(0-1).$$

$$= -i-j-k.$$

Let, $\phi = x+z.$

$$\nabla\phi = i+k$$

$$\therefore \bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i+k}{\sqrt{2}}.$$

$$\therefore \text{curl}\cdot\bar{F}\cdot\bar{N} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$$

$$= \int_S \text{curl } \bar{F}\cdot\bar{N}\,ds$$

$$= \int -\sqrt{2}\,ds = -\sqrt{2}s$$

$$= -\sqrt{2}\,\pi r^2$$

$$= -\sqrt{2}\,\pi \left(\frac{a}{\sqrt{2}} \right)^2$$

$$= \frac{-\pi a^2}{\sqrt{2}}.$$

☆ FOURIER SERIES.

* Periodic function:

→ $\boxed{\begin{array}{l} f(x) = f(x+T) = f(x+2T) = \cdots \\ \text{period} = T \end{array}}$

* Trignometric Series:-

→ A functional series of the form

$$\boxed{a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + \infty.}$$

is said to be trignometric series.

* Fourier Series:

→ Let, $f(x)$ be a periodic f^n define in $[c, c+2l]$ with period $2l$ then the fourier series of $f(x)$ is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]}$$

where, a_0, a_n, b_n are fourier co-efficient given by,

$$\boxed{a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\, dx.}$$

$$\boxed{a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx}$$

$$\boxed{b_n = \frac{1}{l} \int_c^{c+2l} f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx.}$$

NOTE: $[-l, l]$, $[0, 2l]$, $[-\pi, \pi]$, $[0, 2\pi]$

$\downarrow$ $c = -l$ $\quad$ $\downarrow$ $c = 0$ $\quad$ $\downarrow$ $c = -\pi$, $l = \pi$ $\quad$ $\downarrow$ $c = 0$, $l = \pi$

* Dirichlet's Conditions:

→ A function $f(x)$ is said to satisfy Dirichlet's conditions if

(i) $f(x)$ and its integrals are finite & single value.

(ii) $f(x)$ has finite no. of finite discontinuities.

(iii) $f(x)$ has finite no. of maxima & minima.

NOTE: This conditions are the sufficient condⁿ but not necessary to write a fourier series expansion.

* Convergence :-

→ (1) If $f(x)$ is cont. at $x = c \in (a, b)$ then the fourier series of $f(x)$ at $x = c$ convergence to $f(c)$.

then the fourier series of $f(x)$ at $x=c$

convergece to $\frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$.

(3) The fourier series of $f(x)$ at the end points

i.e. at $x=a$ or at $x=b$ convergece to

$$\frac{1}{2}\left[\lim_{x\to a^+} f(x) + \lim_{x\to b^-} f(x)\right].$$

* Fourier Series of Even and odd functions in $[-l,l]$ (OR) $[-\pi,\pi]$:-

→ The fourier series of an odd f^n $f(x)$ in the $[-l,+l]$ is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}$$

where,

$$\boxed{b_n = \frac{2}{l}\int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right) dx}.$$

→ The fourier series of an even f^n $f(x)$ in the $[-l,+l]$ is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)dx$, $a_n = \frac{2}{l}\int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$

* Half - Range Series:

→ ① Half - Range Sine series in $[0, l]$ is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).}$$

$$\boxed{\text{where, } b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx.}$$

→ ② The Half - Range Cosine series of $f(x)$ in $[0, l]$ is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).}$$

$$\boxed{\text{where, } a_0 = \frac{2}{l}\int_0^l f(x)\, dx.}$$

$$\boxed{a_n = \frac{2}{l}\int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx}$$

Ex-1 If $f(x) = \begin{cases} \ldots \\ 1, & 0 < x < 2 \end{cases}$ then ... constant term in the fourier series of $f(x)$ is ___.

(A) 0 (B) 1 (C) 1/2 (D) 2.

Ans: $(-2, 2)$, $l = 2$.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2}\int_{0}^{l} 1\,dx = 1.$$

$\therefore$ Const. term $= \frac{a_0}{2} = \frac{1}{2}$.

Ex-2 If $f(x) = \begin{cases} -\cos x, & -\pi < x \leq 0 \\ \cos x, & 0 \leq x \leq \pi \end{cases}$ then the fourier series of $f(x)$ has the following terms in the expansion.

(A) cosine only (B) sine term only

(C) both cosine & sine terms

(D) None.

Ans:
$$f(-x) = \begin{cases} -\cos x, & -\pi \leq -x \leq 0 \\ \cos x, & 0 \leq -x \leq \pi. \end{cases}$$

$$= \begin{cases} -\cos x ; & 0 \leq x \leq \pi \\ \cos x ; & 0 \geq x \geq -\pi \end{cases}$$

$$= -f(x)$$

So, odd f^n and only sine terms.

Ex-3 Co-coefficient ... series expansion of $f(x) = \begin{cases} -x+1, & -\pi \le x \le 0 \\ x+1, & 0 \le x \le \pi. \end{cases}$

(a) $\sum \frac{(-1)^n}{n^2}$ (b) $\frac{1}{n^2}$ (c) $\frac{\pi^2}{6}$ (d) 0. ✓

Ans: $f(x) = |x| + 1$ in $[-\pi, \pi]$.

So, even f^n and only cosine terms and co-efficient is 0.

Ex-4 If $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ the coeff. of $\cos\left(\frac{n\pi x}{2}\right)$ is ——.

Ans: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right]$

$$\therefore a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right).$$

$$\therefore a_n = \frac{1}{2} \int_{0}^{2} x \cdot \cos\left(\frac{n\pi x}{l}\right) . dx$$

$$a_n = \frac{1}{2} \left[x \cdot \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(- \frac{\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) \right]_0^2$$

$$= \frac{1}{2} \left[0 + \frac{(2)^2 (-1)^n}{(n\pi)^2} - \frac{(2)^2}{(n\pi)^2} \right].$$

$$= \frac{1}{2} \left[- \frac{\cancel{2}}{\cancel{n\pi}} - \frac{4}{(n\pi)^2} \right]. = \cancel{-\frac{1}{n\pi}} \left[1 + \frac{2}{n\pi} \right].$$

$$I = \frac{1}{2}\left[\frac{\ldots}{(n\pi)^2} \quad \frac{\ldots}{(n\pi)^2} \right].$$

$$= \frac{2}{(n\pi)^2}\left[(-1)^n - \frac{1}{\cancel{n\pi}} \right].$$

$$\boxed{ba_n = \frac{2}{(n\pi)^2}\left[(-1)^n - 1 \right].}$$

OM
5

Sujal Patel

ECE

Maths (Complex Variable).

ACE Academy.

PM I (B).

Complex

* Complex Number:

→ A no. of the form $\boxed{z = x + iy}$ is called Complex no. where x & y are real nos.

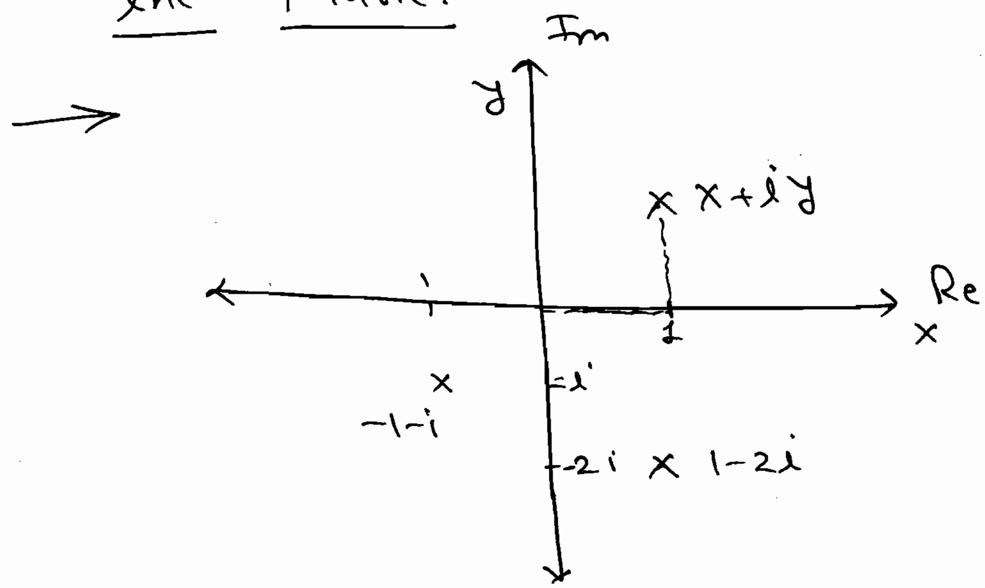
x is called Real part &

y is called Imaginary part &

$\boxed{i = \sqrt{-1}}$

⇒ Representation of the Complex no. in the Plane:

→



⇒ Conjugate of a Complex no.

→ if $z = x+iy$ then Conjugate of z is given by $\boxed{\bar{z} = x - iy.}$

⇒ Modulus of a Complex no.

→ if $z = x+iy$ then ~~conjugate~~ &

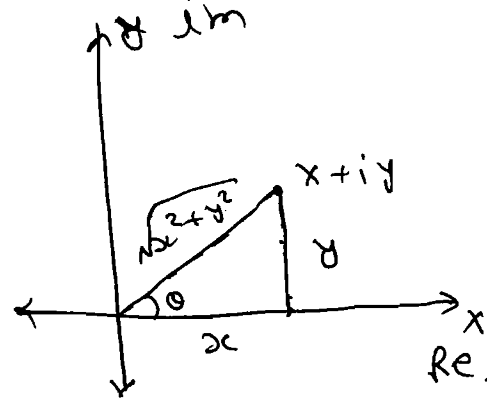
$\boxed{|z| = \sqrt{x^2+y^2}}$ is a real no.

⇒ Polar form of a Complex no.

→ $z = x + iy$.

$\therefore$ $\boxed{x = r\cos\theta}$ + i

$\boxed{y = r\sin\theta.}$



$\therefore$ $z = r\cos\theta + ir\sin\theta$

$z = r\,[\cos\theta + i\sin\theta]$

$\therefore$ $\boxed{z = re^{i\theta}}$

$\therefore$ $r = \sqrt{x^2+y^2}$ = Modulus

θ is argument or amplitude

$\theta = \tan^{-1}(y/x)$.

Ex-1 if $z = x + iy$ then find $|e^{iz}|$.

Ans: $|e^{iz}| = e^{|i(x+iy)|}$

$= e^{|xi - y|}$.

$= |e^{-y}| \cdot |e^{ix}|$

$= e^{-y} \cdot |[\cos x + i\sin x]|$

$= e^{-y} \cdot \sqrt{\cos^2 x + \sin^2 x}$

$= e^{-y}$.

$$= \frac{\quad}{3+4i}$$

<u>Ans</u>: $\dfrac{-5+10i}{3+4i}$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{|-5+10i|}{|3+4i|}$$

$$= \frac{\sqrt{25+100}}{\sqrt{9+16}}$$

$$= \frac{\cancel{5}\sqrt{5}}{\cancel{5}}$$

$$= \sqrt{5}.$$

<u>Ex-3</u> find i^i

<u>Ans</u>: $i = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$

Now, $i^i = \cancel{\cos\frac{\pi}{2}} \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\pi/2}.$

<u>Ex-4</u> $(-i)^i$

Now, $-i = e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i.$

$\therefore (-i)^i = \left(e^{-i\frac{\pi}{2}}\right)^i = e^{\pi/2}.$

<u>Ex-5</u> $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ find z^4. ?

<u>Ans</u>: $z = \frac{\sqrt{3}}{2} + \frac{i}{2} = e^{\frac{i\pi}{6}}$

Now, $z^4 = \left(e^{+i\frac{\pi}{6}}\right)^4 = e^{i\frac{2\pi}{3}}$

$$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

$$= -\frac{1}{2} + i\sqrt{3}/2.$$

Ex-6 find $(\sqrt{3}+i)$

$\therefore \quad z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) 2$

$z = 2 \cdot e^{i\frac{\pi}{6}}.$

$\therefore \quad \cancel{(\sqrt{3}+i)^7 = 2^7 \cdot \left(-\frac{1}{2} + i\sqrt{3}/2\right)}.$

$z^7 = 2^7 \cdot e^{i\frac{7\pi}{6}}.$

$= 2^7 \cdot \left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right].$

$= 2^7 \cdot \left[-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right].$

$= 2^7 \cdot \left[-\frac{\sqrt{3}}{2} - \frac{i}{2}\right].$

$\therefore \boxed{z^7 = -2^6 \cdot [\sqrt{3} + i]}.$

Ex-7 The value of expression $\frac{-5+10i}{3+4i}$.

Ans: $z = \frac{-5+10i}{3+4i} \times \frac{3-4i}{3-4i}.$

$= \frac{-15+20i+30i+40}{9+16}$

$= \frac{25+50i}{25}$

$= 1+2i$

→ Corresponding the each $\overset{\text{comp.}}{}$ variable z in the region R of z plane there corresponds a unique complex no. w in the Region R' of w plane then w is called a complex function.

→ $w = f(z) = u(x,y) + i\,v(x,y).$

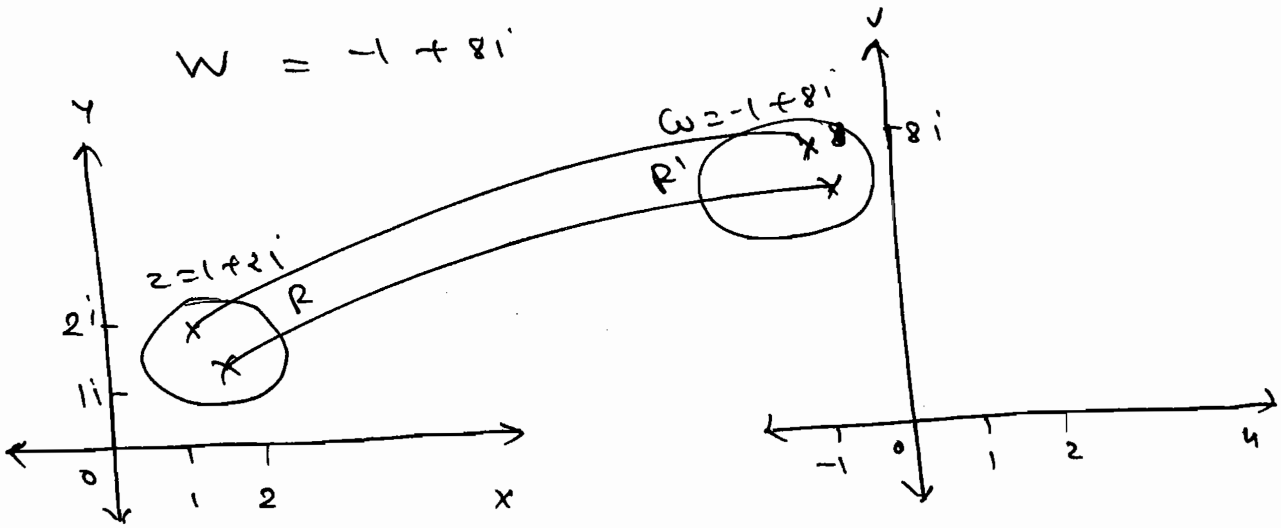
e.g. $w = f(z) = z^2 + 2z.$

$z = 1+2i.$

$\therefore \quad f(z) = (1+2i)^2 + 2(1+2i)$

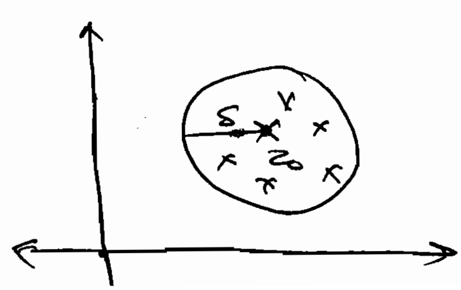
$= 1 + 4i - 4 + 2 + 4i$

$w = -1 + 8i$



* Neighbourhood of point z_0 :-

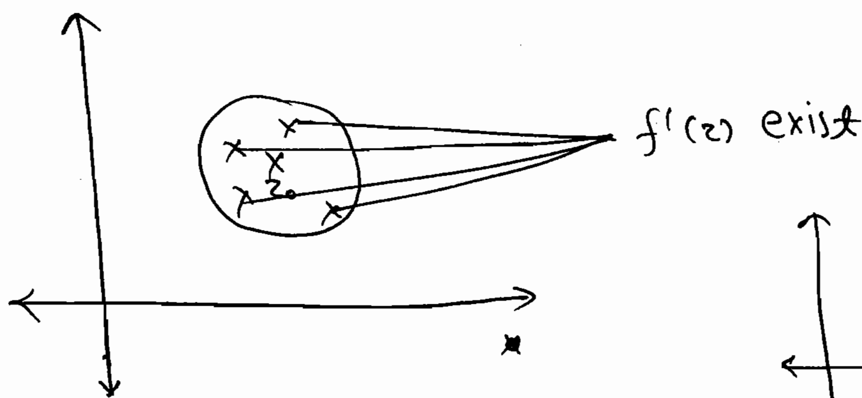
→ Set of all point lies inside the circle with centre z_0 and radius δ is called δ neighbourhood of point z_0.

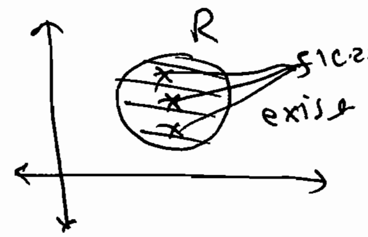


$\Rightarrow \boxed{|z - z_0| < \delta}$

☆ Analytic function:

→ A function $f(z)$ is said to be a analytic at a point if $f'(z)$ is exist not only the point z_0 but also in some neighbourhood of the point.





→ A function $f(z)$ is said to be analytic in the Region if $f'(z)$ exist at every point of the region.

☆ Entire function:

→ A function $f(z)$ is said to be an entire f^n if it is anaytic throughout the finite Complex plane.

e.g. → $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$

→ $f(z) = \cos z$.

→ $f(z) = \sin z$.

→ $f(z) = e^z$.

* Singularity:

→ A point at which the f^n is not analytic is called singularity.

e.g. $f(z) = \frac{z^2+4}{z^2+9}$.

$z^2+9=0 \Rightarrow z = \pm 3i$.

are singular point.

e.g. $f(z) = \sqrt{z}$.

$f'(z) = \frac{1}{2\sqrt{z}}$.

$z=0$ is singular point.

→ $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$ is entire f^n.

Note: if $f(z)$ & $g(z)$ is two entire f^n then

(i) $f(z) \cdot g(z)$ is also entire f^n.

(ii) $f(z) \pm g(z)$ is also entire f^n.

(iii) $\frac{f(z)}{g(z)}$ is also entire f^n. $(g(z) \neq 0)$

* Harmonic function:

→ A f^n $f(x,y)$ is said to be a Harmonic f^n if it is satisfies the laplace $\frac{}{}$eqn.

i.e. $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

$\therefore \boxed{f_{xx} + f_{yy} = 0.}$

$\Rightarrow$ $f(z)$ is said to be analytic if it satisfied following two condn.

$f(z) = u(x,y) + i\,v(x,y)$.

Analytic $\Longleftrightarrow$ (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, exist

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\therefore$ $\boxed{\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}}$

* C.R. eqn in polar form:

$\rightarrow$ $f(z) = u(r,\theta) + i v(r,\theta)$.

condn, $\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\,\frac{\partial v}{\partial r}$.

i.e. :- $\boxed{\begin{matrix} u_r = \frac{1}{r}\cdot v_\theta \\ u_\theta = -r\cdot v_r. \end{matrix}}$

$\boxed{\text{Note:}}$ $\rightarrow$ $f(z) = u(x,y) + i v(x,y)$ is an analytic fn then the curves

(i) $\boxed{u(x,y) = c_1 \;\&\; v(x,y) = c_2 \text{ are orthogonal to each other.}}$

$\rightarrow$ $f(z) = u(x,y) + i v(x,y)$ is an analytic fn then u & v are, harmonic conjugate of each other.

① Real part is given (i.e. $u(x,y)$ is given).

$$\therefore \boxed{f(z) = \int u_x(z,0)\,dz \mp i \int u_y(z,0)\,dz + C.}$$

$$\therefore v = -\int_{y \to \text{constant}} \frac{\partial u}{\partial y}\,dx + \int \left(\text{terms of } \frac{\partial u}{\partial x} \text{ not containing } x\right) dy + C.$$

② Imaginary part is given (i.e. $v(x,y)$ is given).

$$\therefore \boxed{f(z) = \int v_y(z,0)\,dz + i \int v_x(z,0)\,dz + C.}$$

$$\therefore u = \int_{y \to \text{constant}} v_y\,dx - \int (\text{terms } v_x \text{ without } x)\,dy + C$$

Ex-1 If $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ is not analytic at

(a) $(1,0)$ (b) $(0,1)$ (c) $(0,0)$ (d) $(2,0)$.

Ans: $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$.

~~$f'(z) = \frac{1}{2(x^2+y^2)}$~~

Ex-2 $f(z) = e^{-x}(\cos y - i\sin y)$.

Ans: $f(z) = e^{-x}\cos y - i e^{-x}\sin y$.

$\rightarrow u = e^{-x}\cos y, \quad v = -e^{-x}\sin y.$

$u_x = -e^{-x}\cos y, \quad v_x = e^{-x}\sin y.$

$u_y = -e^{-x}\sin y, \quad v_y = -e^{-x}\cos y.$

$u_x = v_y$ & $u_y = -v_x$

So, C.R. eqn satisfy and f^n is analytic at every point.

and also it is entire f^n.

Ex-3 $f(z) = 3xy + i(x^3 - y^3)$.

Ans: $u = 3xy$ $\qquad v = x^3 - y^3$.

$\therefore u_x = 3y$ $\qquad v_x = 3x^2$

$\therefore u_y = 3y$ $\qquad v_y = -3y^2$.

$u_x \neq v_y, \quad v_x \neq -u_y$.

So, not analytic f^n & not entire f^n.

Ex-4: $f(z) = u + iv$ is analytic?

(a) $iu_x + v_x = u_y + iv_y$.

(b) $iu_x + v_x = -v_y - iv_y$.

(c) $u_x + iv_x = -iu_y + v_y$.

(d) $u_x + iv_y = iu_y - v_y$.

Ex-5 $f(z) = (x + ay) + i(bx + cy)$ is analytic then which of the following is true?

A (a) $c=1, b=1, a=1$ $\qquad$ (d) $c=1, b=1, a=-1$.

(b) $c=1, a=-1, b=2$

(c) $c=1, b=-1, a=-1$

Ans: $u = x + ay$. $\qquad v = bx + cy$.

$\therefore u_x = 1$ $\qquad v_x = b$.

$u_y = a$ $\qquad v_y = c$.

$\boxed{c=1}$ $\qquad a = -b$.

analytic?

Ans: $u = r^2\cos\theta,\quad v = r^2\sin p\theta.$

$\therefore u_r = 2r\cos\theta,\quad v_r = 2r\sin p\theta.$

$\therefore u_\theta = -r^2\sin\theta.\quad v_\theta = pr^2\cos p\theta.$

$\therefore u_r = \frac{1}{r}v_\theta.$

$\therefore 2r\cos^2\theta = \frac{1}{\cancel{r}}\times p r^{\cancel{2}}\cos 2p\theta$

$\therefore \boxed{p = 2}$

Ex-7 if $u = x^3 - 3xy^2$ then the analytic fn.?

Ans:

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + c.$$

$\therefore$ ~~from~~ $u_x = 3x^2 - 3y^2$ $\qquad z = x+iy$

$u_x(z,0) = 3z^2.$

$u_y = -6xy.$

$u_y(z,0) = 0.$

$$\therefore f(z) = \int 3z^2.dz - i\int 0.dz + c.$$

$\therefore f(z) = z^3 + c.$

$\therefore f(z) = (x+iy)^3 + c.$

Ex-8 Find the Analytic function $f(z) = u+iv$, where $v = e^x(\cos y + \sin y)$.

Ans: $v_x = e^x(\cos y + \sin y). \rightarrow v_x(z,0) = e^z$

$v_y = e^x[-\sin y + \cos y]\quad v_y(z,0) = e^z.$

$\therefore \quad f(z) = +\int V_y(z,0)\,dz + i\int V_x(z,0) + c.$

$$= \int e^z.dz + i\int e^z.dz + c.$$

$$= e^z + ie^z + c$$

$$\therefore \boxed{f(z) = e^z(1+i) + c}$$

Ex-7 If $u = \log(x^2+y^2)$ then find V so that $f(z) = u+iv$ is analytic.

Ans: $u_x = \dfrac{1}{x^2+y^2}.2x$

$u_y = \dfrac{1}{x^2+y^2}.2y.$

$\therefore \quad v = -\int u_{\cancel{y}}.dx + i\int (0).dy + c.$

$V = -\int \dfrac{2y}{x^2+y^2}.dx. + c.$

$V = -\dfrac{2y}{y}\tan^{-1}(x/y). + c.$

$\therefore V = -\cancel{i}2\cancel{y}\tan^{-1}(x/y) + c.$

Ex-8 If $V = e^x \sin y$ then find orthogonal trajectory of V.

Ans: $V_x = e^x \sin y, \quad V_y = e^x \cos y.$

$\therefore \quad u = u_x\,dx \cancel{+u_y}\,dy.$

$\therefore u = +V_y\,dx \mp V_x\,dy. + c.$

$\therefore u = +\int e^x.\cos y\,dx \mp 0 + c$

Ex-8 If $u = 3x^2 - 3y^2$ then find v so that $f(z) = u + iv$ is analytic.

Ans: $u_x = 6x$, $u_y = -6y$.

$\therefore v = v_x \cdot dx + v_y \cdot dy$.

$= -u_y \cdot dx + u_x \cdot dy$.

$= -\int(-6y)\, dx + 0. + c$

$\therefore \boxed{v = 6xy + c}$

Ex-9 $u = xy$ then find v where $f(z) = u + iv$ is analytic.

Ans: (a) $\frac{(x+y)^2}{2} + k$ (b) $\frac{x-y^2}{2} + k$

(c) $\frac{y^2 - x^2}{2} + k$ (d) $\frac{(x-y)^2}{2} + k$.

Ans: $u_x = y$, $v_x = x$.

$\therefore v = v_x \cdot dx + v_y\, dy$.

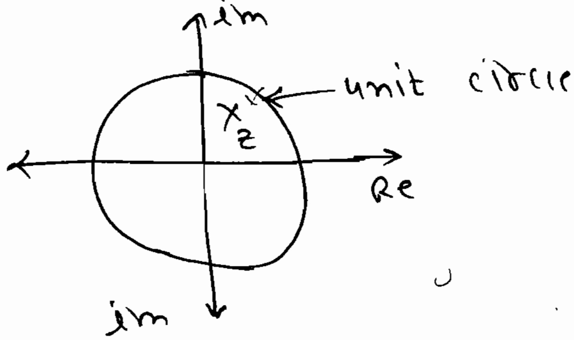
$= -u_y \cdot dx + u_x \cdot dy$.

$= -\int x \cdot dx + \int y \cdot dy + c$.

$= -\frac{x^2}{2} + \frac{y^2}{2} + c$

$\therefore \boxed{v = \frac{y^2 - x^2}{2} + c}$

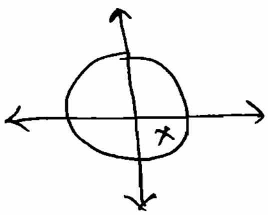
Ex-11 A point z has been ploted in the complex plane as shown in the figure. then the plot of the complex no. is $y = 1/z$ is.



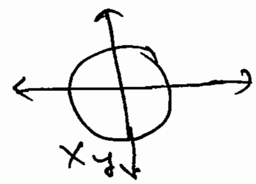
$0 \leq x, y \leq 1/\sqrt{2}$

$0 \leq x, y \leq 0.707$

(a)



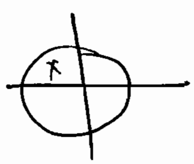
(b)



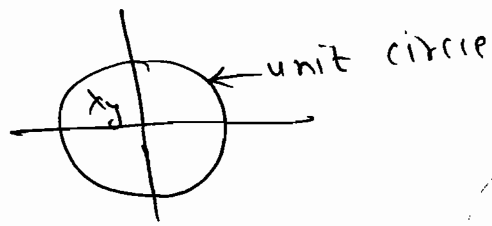
$x + iy$

$\sqrt{x^2+y^2} < 1$

(c)



(d) ✓



$\sqrt{\frac{1}{2} + \frac{1}{2}}$

$|z| < 1$ & $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$u, v > 0.$ $\frac{1}{2} + \frac{1}{2}$

$$y = \frac{1}{z}$$

$$y = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}.$$

$|z| < 1$ (1)

$$\therefore \boxed{y = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}}$$

let, $z = 0.8 + i0.8.$

$$\therefore y = \frac{0.8}{0.64+0.64} - i\frac{0.8}{0.64+0.64}.$$

$$y = \frac{0.8}{1.28} - i\frac{0.8}{1.28}$$

which lies inside the unit circle.

$$\therefore \left|\frac{1}{y}\right| = |z| < 1.$$

$$\therefore \left|\frac{1}{y}\right| < 1$$

$$\therefore \boxed{|y| > 1.}$$

* Complex Integration

→ Evaluation of an integration of a function along a continuous curve is called complex integration and is given by.

$$\int_C f(z)\,dz.$$

Ex-1 Evaluate $\int_1^{1+i\pi} e^z\,dz.$

Ans:

$$I = \left[e^z\right]_1^{1+i\pi}$$

$$= e^{1+i\pi} - e^1$$

$$= e^1\left[\left[\cos\pi + i\sin\pi\right] - 1\right]$$

$$= e\left[-1-1\right].$$

$$= -2e.$$

Ex-2 Evaluate $\int_0^{2\pi} z^2 \sin 4z \cdot dz$

Ans:

$$I = \left[(z^2)\left(-\frac{\cos 4z}{4}\right) - (2z)\left(-\frac{\sin 4z}{16}\right) + (2)\left(\frac{\cos 4z}{64}\right).\right]_0^{2\pi}$$

$$= \left[(2\pi)^2\left(-\frac{1}{4}\right) - 0 + \cancel{(2)\left(\frac{1}{64}\right)} - (0) + 0 - \cancel{\frac{2}{64}}\right].$$

$$= -\pi^2.$$

Ex-3 Evaluate $\int_0 (x^2 - iy)\, dz$ along $y = x^2$

Ans: Here, $y = x^2$

$$\therefore dy = 2x\, dx$$

$$dz = dx + i\, dy.$$

$$dz = dx + 2xi\, dx$$

$$= (1+2xi)\, dx.$$

$$I = \int_0^1 (x^2 - i(x^2))\,(1+2xi)\, dx.$$

$$= \int_0^1 (x^2 + 2x^3 i - ix^2 + 2x^3)\, dx.$$

$$= \int_0^1 2x^3 + x^2 + (2x^3 - x^2)\, i.$$

$$= \left[\frac{x^4}{2} + \frac{x^3}{3} + \left(\frac{x^4}{2} - \frac{x^3}{3}\right) i\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{2} - \frac{1}{3}\right) i$$

$$= \frac{5}{6} + \frac{1}{6} i$$

$$= \cancel{\frac{7+2i}{12}} = \frac{5+i}{6}.$$

Ex-4 find the value of the integral $\int_c$ Real part of $(z)\, dz$. Where c is the shortest path joint the point $1+i$ to $3+2i$.

Ans: $\cancel{I = \int_1^3 x\, dx = \left[\frac{x^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = 7/2.}$

$(1,1) \qquad (3,2).$

$$\therefore \cancel{\frac{y_2-y_1}{y-y_1}} \; \cancel{=} \; \cancel{\frac{x_2-x_1}{}}$$

$$\therefore \frac{y_2-y_1}{y-y_1} = \frac{x_2-x_1}{x-x_1}.$$

$$\therefore \frac{1}{y-1} = \frac{2}{x-1}.$$

$$x-1 = 2y-2.$$

$$\therefore x-2y = -1.$$

$$\therefore dx - 2dy = 0$$

$$dx = 2dy.$$

$$\therefore I = \int_0^1 x\,(dx+i\,dy).$$

$$= \int_0^1 x\left(dx + i\frac{dx}{2}\right).$$

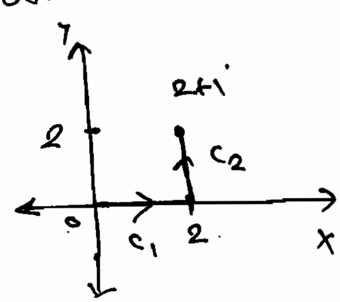
$$= \int_0^1 x\left(1+\frac{i}{2}\right)dx$$

$$= \left(1+\frac{i}{2}\right)\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left(1+i/2\right)$$

$$\therefore I = \frac{2+i}{4}.$$

Ex-5 Evaluate $\int_0^{2+i} (\bar{z})\,dz.$ along the Real axis to '2' and the to $2+i$.



$\therefore y=0 \Rightarrow dy=0$

$x=2. \qquad dx=0$

$$\cancel{I_1 = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2.}$$

$$I = \int_0^{2+i} (\bar{z})^2 = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz.$$

(i) <u>along</u> <u>OA</u>:

$y = 0 \Rightarrow dy = 0$ & x: '0' to '2'.

$$\therefore I_1 = \int_{OA} (\bar{z})^2 dz = \int_{OA} (x - iy)^2 (dx + i dy).$$

$$= \int_0^2 x^2 \cdot dx$$

$$= \left[x^3/3 \right]_0^2$$

$$I_1 = 8/3$$

(ii) along AB:

$x = 2 \Rightarrow dx = 0$: y : 0 to 1.

$$\therefore I_2 = \int_{AB} (\bar{z})^2 dz = \int_{AB} (x - iy)^2 \cdot (i dy).$$

$$= \int_{AB} (2 - iy)^2 \cdot (i dy).$$

$$= \int_0^1 (4 - i2y - y^2)\, i\, dy.$$

$$= i \left[4y - iy^2 - y^3/3 \right]_0^1 = i \left[4 - i - 1/3 \right]$$

$$= [\ldots 4 - i\,]\, i = 4i + 1.$$

$$I = \frac{8}{3} + \frac{11}{3}i + 2$$

$$\therefore I = \frac{14+11i}{3}$$

NOTE: here, $\boxed{\bar{z}}$ is not analytic so we can't take direct path OB but if it is z^2 instead of $(\bar{z})^2$ we can take path OB direct.

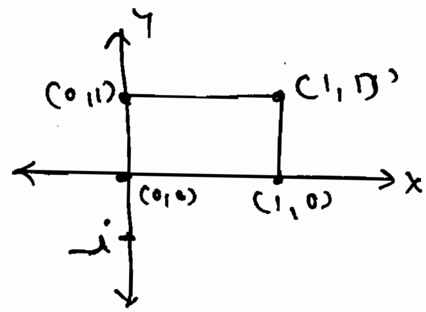
i.e. $$\int_{OA} z^2\,dz + \int_{AB} z^2\,dz = \int_{OB} z^2\,dz.$$

☆ Cauchy's Integral Theorem:

→ Let $f(z)$ is analytic within and on the closed region bounded by closed curve C, then $\boxed{\int_C f(z)\,dz = 0.}$

e.g. (i) $\int_C z^2\,dz = 0.$

'c' is OABC



(ii) $\int_C \frac{z^2}{z+i}\,.\,dz = 0.$

(iii) $\int_C \frac{z^2}{(z-(\frac{1}{2}+i/2)} = 0.$

(iv) $\int_C \frac{z^2}{z-(\frac{1}{2}+i/2)}\,dz \neq 0.$

because $f(z)$ is not analytic at $z = \frac{1}{2} + i/2$ within the bounded closed curve

* Cauchy's Integral Formula:

→ Let, $f(z)$ is analytic within and on a closed curve C and $z=a$ is any point inside the curve C then

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\cdot f(a).}$$

e.g.

$$\int_C \frac{(z^2)}{z-\left(\frac{1}{2}+i/2\right)}\,dz.$$

$f(z) = z^2$

$z = a = \frac{1}{2} + \frac{i}{2}.$

$$\therefore \int_C \frac{(z^2)}{\left(z-\left(\frac{1}{2}+i/2\right)\right)}\,dz = 2\pi i\cdot\left(\frac{1}{2}+i/2\right)^2$$

$$= 2\pi i\left[\cancel{\tfrac{1}{4}} + \frac{i}{2} - \cancel{\tfrac{1}{4}}\right]$$

$$= \pi i^2$$

$$= -\pi.$$

NOTE:

$$\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\cdot f(a).$$

diffⁿ w.r.t. a

$$\int_C \frac{(-1)\,f(z)}{(z-a)^2}\cdot(-1)\,dz = 2\pi i\cdot f'(a).$$

again

$$\int 2\,\frac{f(z)}{(z-a)^3}\cdot dz = 2\pi i\cdot f''(a).$$

$\vdots$

$$\therefore \boxed{\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!}\, f^{(n-1)}(a).}$$

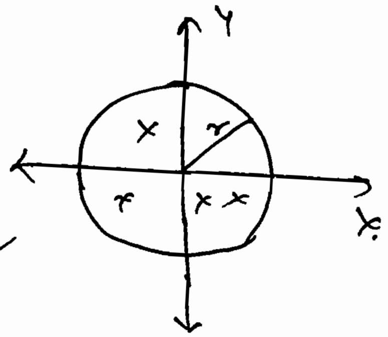
Ex-1 Evaluate $\int_C \frac{z^2 - z + 1}{(z-1)}\,dz$ where c is $|z+1| = 2.5$.

Ans:

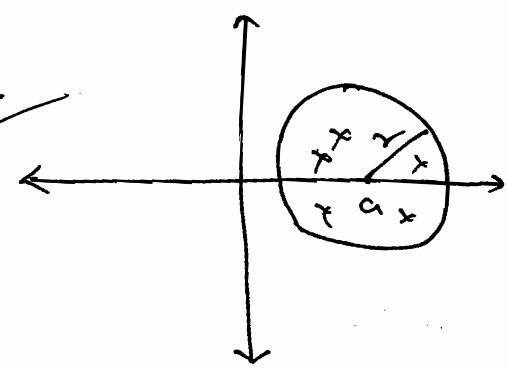
$\cancel{-2.5 < z+1 < 2.5}$

$\cancel{-3.5 < z < 3.5}.$

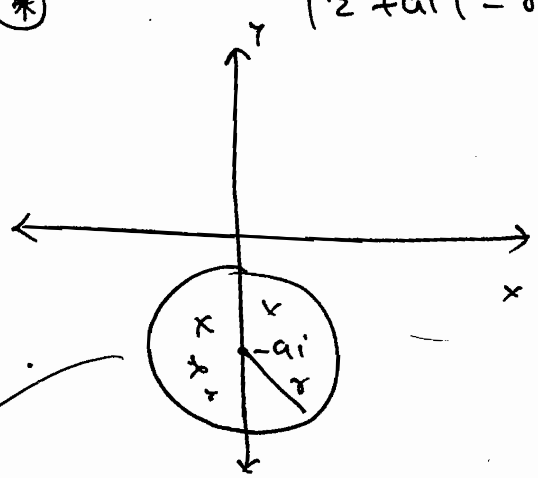
(*) $|z| = r$



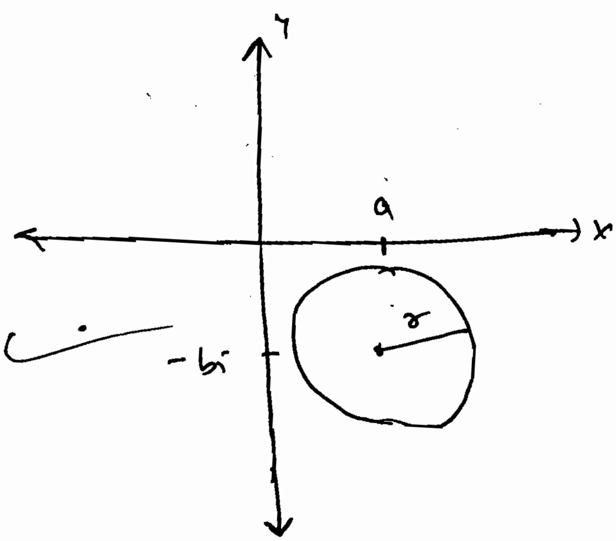
(*) $|z-a| = r$



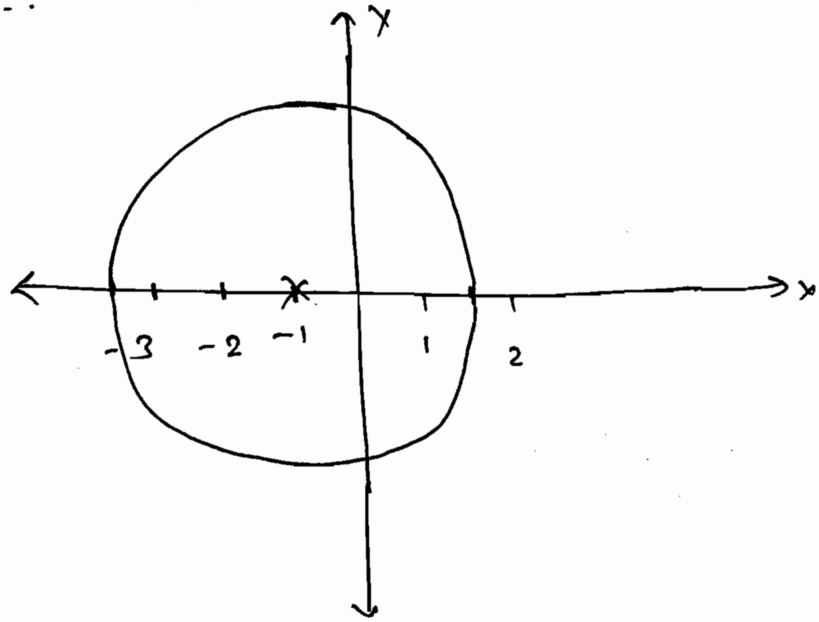
(*) $|z+ai| = r$



(*) $|z-a+bi| = r$



$\therefore \quad |z+1| = 2$



$$= 2\pi i \cdot f(1)$$

$$= 2\pi i \cdot (1-1+1)$$

$$= 2\pi i$$

Ex-2 Find $\int \frac{e^{2z}}{(z-1)(z-2)} \cdot dz$ where 'c' is $|z| = 3$

Ans: here, $z = 1, 2$ are into $|z| = 3$.

$$I = \int \frac{e^{2z}}{(z-1)(z-2)} \cdot dz$$

$$= \int \frac{e^{2z}}{(z-2)} - \frac{e^{2z}}{(z-1)} \, dz.$$

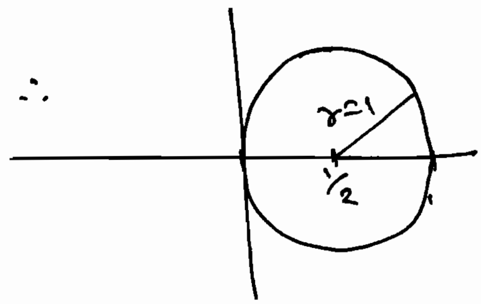
$$= 2\pi i \cdot f(2) - 2\pi i \cdot f(1).$$

$$= 2\pi i \left[e^{4} - e^{2} \right].$$

Ex = Find $\int_c \frac{e^{2z}}{(z-1)(z-3)} \cdot dz$

Ans: $|z-\frac{1}{2}| = 1.$

$\therefore$



$z=1$ lies inside &
$z=3$ lies outside.

$$\therefore I = \frac{1}{2}\int_c \frac{e^{2z}}{(z-3)} - \frac{1}{2}\frac{e^{2z}}{(z-1)} \cdot dz.$$

$$= 0 - \frac{1}{2}\int \frac{e^{2z}}{(z-1)} \cdot dz.$$

$$= -\frac{1}{2} \times 2\pi i \cdot f(1).$$

$$= -\pi i \cdot e^2$$

$$I = -\pi e^2 i.$$

(or)

$$\therefore I = \int_c \frac{\frac{e^{2z}}{(z-3)}}{(z-1)} \cdot dz.$$

$$f(z) = \frac{e^{2z}}{(z-3)}.$$

$$= 2\pi i \cdot f(1).$$

$$= 2\pi i \cdot \frac{e^2}{(-2)}$$

$$\therefore I = -\pi e^2 \cdot i.$$

Ex-4 find $\int_c \frac{1}{e^z \cdot z^2} \cdot dz$ where c is a simple closed curved around the origin.

Ans: $I = \int_c \frac{\frac{1}{e^z}}{z^2} \cdot dz$

$z=0$ is lies inside

$\therefore I = \frac{2\pi i}{1!} \cdot f'(0)$.

$f(z) = \frac{1}{e^z}$.

$= -2\pi i \cdot \left(e^{\frac{1}{z}}\right)$

$f'(z) = -e^{-z} = -\frac{1}{e^z}$

$= -2\pi i \cdot e^{\frac{1}{0}}$

$f'(0) = -1$.

$\therefore I = -2\pi i$.

Ex-5 Find $\int_c \frac{\sin^2 z}{(z-\frac{\pi}{6})^3} dz$ where 'c' is unit circle. $|z|=1$.

Ans: $z = \frac{\pi}{6}$ lies is inside $|z|=1$.

So. $I = \int_c \frac{\sin^2 z}{(z-\frac{\pi}{6})^3}$

$I = \frac{2\pi i}{2!} \cdot f''(\frac{\pi}{6})$

Now, $f'(z) = \sin 2z$

$f''(z) = 2\cos 2z$

$f''(\frac{\pi}{6}) = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$.

$\therefore \boxed{I = \pi i}$

$$f(a) = \int \frac{2z^2 - z - 2}{z - a} \, dz. \quad \text{and 'c' is } |z| = 5/2.$$

Ans:

① $$f(2) = \int \frac{2z^2 - z - 2}{(z-2)} \, dz$$

$z = 2$ is lies inside $|z| = 2.5$.

So, $f(2) = 2\pi i \cdot f(2)$.

$$= 2\pi i \cdot [8 - 2 - 2]$$

$$\boxed{f(2) = 8\pi i.}$$

$$f(3) = \int \frac{2z^2 - z - 2}{(z-3)} \, dz$$

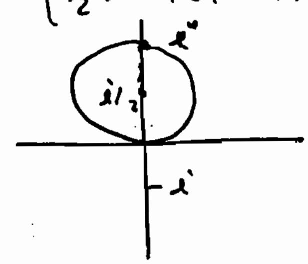
$z = 3$ outside $|z| = 2.5$

So, $\boxed{f(3) = 0}$

Ex-7 Find $\int_c \frac{dz}{z^2+1}$ where c is $|z - i/2| = 1$.

Ans: $$I = \int \frac{dz}{(z+i)(z-i)}.$$

$\therefore$ $|z - i/2| = 1.$



$$\therefore \quad I = \int \frac{\left(\frac{1}{z+i}\right)}{(z-i)} \cdot dz$$

$z = i$ lies inside the circle.

So, $I = 2\pi i \cdot f(i)$.

$$= 2\pi i \cdot \frac{1}{2i}$$

$\boxed{I = \pi}$

Ex-9 Find $\int_C f(z)dz$ along the unit circle ... the complex plane $f(z) = \frac{\cos z}{z}$.

Ans: $I = \int \frac{\cos z}{z} \cdot dz$

$z=0$ lies inside $|z|=1$

$I = 2\pi i \cdot f(0)$.

$= 2\pi i \cdot \cos(0)$

$\therefore \boxed{I = 2\pi i}$

Ex-10 Evaluate $\int_C \frac{\cos 2\pi z}{(2z-1)(z-3)} dz$ where $|z|=1$.

Ans: $z = 1/2$ lies inside

$z = 3$ outside.

So, $I = \int \frac{\frac{1}{2}\frac{\cos 2\pi z}{(z-3)}}{(z-\frac{1}{2})}$.

$I = \frac{1}{2} \times 2\pi i \times \frac{\cos 2\pi \times \frac{1}{2}}{(\frac{1}{2}-3)}$

$I = - \frac{\pi i}{-5} \times 2$

$\therefore \boxed{I = \frac{2\pi i}{5}}$

$$\oint_c \frac{1+f(z)}{z}. \quad \text{where 'c' is } |z|=1.$$

Ans:

$$I = \int \frac{1+c_0+c_1 z^{-1}}{z} . dz$$

$$I = \int \frac{z+z c_0+c_1}{z^2} .dz$$

$f'(z) = 1+c_0.$

$f'(0) = 1+c_0.$

$$I = \frac{2\pi i}{1!} . f'(0).$$

$$\therefore I = 2\pi i . (1+c_0).$$

Ex - 12 Find $\int_c \frac{z^2+8}{0.5z-1.5j} dz$ with is $x^2+y^2=16$

$\therefore |z|=4.$

Ans:

$$I = \int_c \frac{z^2+8}{\frac{1}{2}(z-3j)} dz$$

$$= \int_c \frac{2(z^2+8)}{(z-3j)} . dz$$

$\therefore z=3$ lies inside $|z|=4.$

$$I = 2\pi i . f(3j).$$

$$= 2\pi j . 2(\cancel{9}\cancel{+8}) . (-2)$$

$$= \cancel{68\pi j}.$$

$$\therefore \boxed{I = -4\pi j}$$

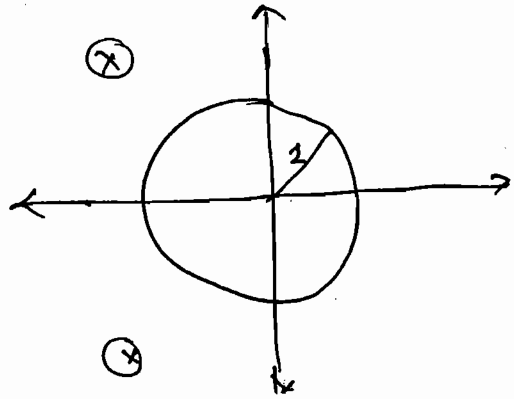
Ex-13 Find $\int_C \frac{\quad}{z^2+2z+5}$ ... $|z|=1$.

Ans: $z^2 + 2z + 1 = -4.$

$\therefore \quad (z+1)^2 = \pm 2i$

$\boxed{z = -1 \pm 2i}$

which is lies outside the $|z| = 1$.

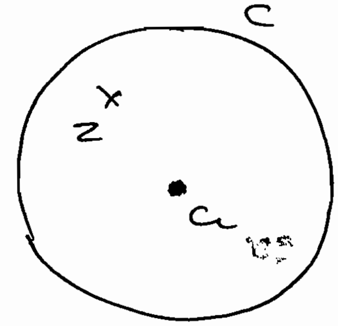


Sol $\boxed{I = 0}$.

Taylor's Series:

→ Let, $f(z)$ is an analytic function inside a circle 'c' with centre 'a', then for any point 'z' inside the circle 'c' then $f(z)$ is as above.

$$\boxed{f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots}$$



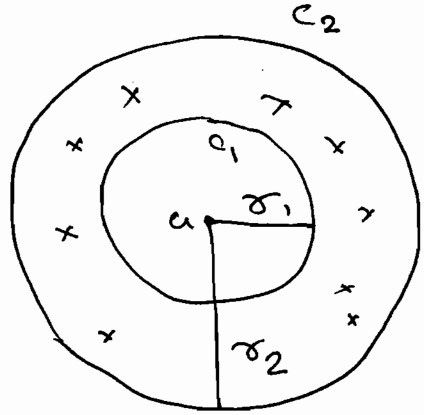
Laurent Series.

→ If $f(z)$ is analytic in a Ring Shaped bounded by two concentretic circle c_1 & c_2 having radius r_1 & r_2 with centre as 'a' then for any point z inside the R.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + a_{-3}(z-a)^{-3} + \cdots$$

$$\therefore \quad f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}}_{\text{principle part}}.$$

where, $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$.



* **Zeros:**

→ A point at which the f^n is zero is called zero of a function.

e.g. $f(z) = \frac{z^2+4}{z^2-4}$.

$z^2+4=0$

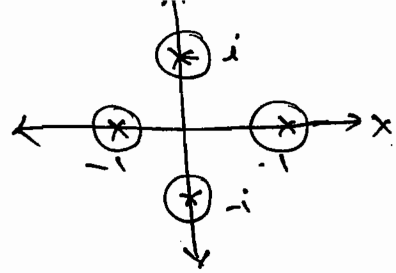
$\therefore$ $\boxed{z = \pm 2i}$ are zeros of f^n.

* **Isolated singularity:**

→ A singularity is said to be an isolated singularity if there exist a neighbourhood at the singularity which doesn't contain any other singularity of the f^n, otherwise it is called non-isolated singularity.

$(z^2+1)(z^2-1)$

so, $z = \pm i, \pm i$



Ex-1 $f(z) = \dfrac{1}{\tan\left(\frac{\pi}{z}\right)}$.

Ans: $\tan\frac{\pi}{z} = 0$.

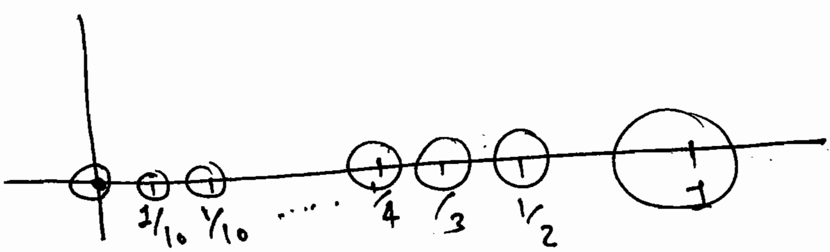
$\therefore \frac{\pi}{z} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, n\pi$

$\frac{\pi}{z} = n\pi$

$\therefore \boxed{z = 1/n.}$ n = integer.

$\therefore z = 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}$.

All point are isolated except $z=0$.



at $z=0$, let $r = 0.0000001$

$z_1 = \dfrac{1}{10000000000} < r$.

which is inside $r = 0.0000001$.

so $z=0$ is not isolated singularity.

* Removable Singularity:

→ In the expansion of the f^n in the form of Laurent's series, if it contains only the terms of analytical part then the singularity is called Removable Singularity.

e.g.: $f(z) = \frac{\sin z}{z}$

$$= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

So, $z=0$ is removable singularity.

* Pole of order one:

→ In the expansion ~~of~~ of the f^n in the form of Laurent's series if the principle part contains terms till $\frac{1}{(z-a)}$ only, then $z=a$ is called pole of order one.

* Pole order n:

→ In the expansion of the f^n in the form of Laurent's series if the principle part contains the terms till

$(z-a)^n$

order n.

i.e. $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$
$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2}$
$+ \cdots + a_{-n}(z-a)^{-n}$.

Ex-1 $f(z) = \frac{z^2}{(z-3)^3}$

$\rightarrow$ $z = 3$ is pole of order '3'.

Ex-2 $f(z) = \frac{\sin z - 1}{z^4}$.

$\rightarrow$ $z = 0$ is pole of order '4'.

Ex-3 $f(z) = \frac{\sin z}{z^2}$.

$$= \frac{z - z^3/3! + \frac{z^5}{5!} - \cdots}{z^2}$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots$$

So, $z = 0$ is pole of order '1'.

Ex-4 $f(z) = \frac{\sin z^2}{z^4}$.

$$\rightarrow = \frac{z^2 - \frac{(z^2)^3}{3!} + \cdots}{z^4}$$

$$= \frac{1}{z^2} - \frac{z^2}{3!} + \cdots$$

So, $z = 0$ is poler of order 2.

* Essential Singularity.

→ In the expansion of the f^n in Laurant's Series if the principle part contain infite no. of terms then the singularity is called Essential Singularity.

i.e.
$$\boxed{f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \infty}$$

e.g. $f(z) = e^{\frac{1}{z-2}}$.

→ $f(z) = 1 + \frac{1}{(z-2)} + \frac{1}{2!\,(z-2)^2} + \frac{1}{3!\,(z-2)^3} + \dots \infty$

so, $z = 2$ is called essential singularity.

Ex-1 $f(z) = \dfrac{e^z}{(z-3)^2(z^2+4)}$.

Ans: $z = 3$ is pole of order 2.

$z = \pm 2i$ is pole of order 1.

Ex-2 $f(z) = \dfrac{1}{\sin z - \cos z}$ at $z = \pi/4$.

Ans: at $\pi/4$.

$f(z) = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$.

So, $z = \frac{\pi}{4}$ is pole of order-1.

Ex - 3 $f(z) = \dfrac{\ldots}{z^4}$ at

Ans: $f(z) = \dfrac{1 - \left[1 + (2z) + \frac{(2z^2)^2}{2!} + \frac{(2z)^3}{3!} + \cdots \infty\right]}{z^4}$

$\therefore$ $f(z) = \dfrac{2}{z^3} + \dfrac{1}{2} + \dfrac{8z^2}{3!} + \cdots$

So, $z = 0$ is pole of order '3'.

Ex - 4 $f(z) = \dfrac{1}{z(e^z - 1)}$ at $z = 0$.

Ans: $f(z) = \dfrac{1}{z(e^z - 1)}$

$$= \frac{1}{z\left[\cancel{1} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \cancel{-1}\right]}$$

$$= \frac{1}{z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \cdots \infty}$$

So, $= \dfrac{1}{z^2\left[1 + \frac{z}{2!} + \frac{z^3}{3!} + \cdots \infty\right]}$

So, $z = 0$ is pole of order '2'.

Ex- 5 $f(z) = \dfrac{z^2-1}{(z+1)(z-1)^3}$.

Ans: $f(z) = \dfrac{\cancel{(z-1)}\cancel{(z+1)}}{\cancel{(z+1)}(z-1)^{\cancel{3}2}}$

$\therefore f(z) = \dfrac{1}{(z-1)^2}$.

So, $z=1$ is pole of order 2.

$z=-1$ is a removable singularity.

Ex- 6 $f(z) = \dfrac{e^{z^2}}{z^3}$ at $z=0$.

Ans: $z=0$ is a pole of order 3.

* Residue:

→ The Co-efficient of $\frac{1}{z-a}$ in the Laurent's expansion of the f^n @ the point $z=a$ is called residue of the function at the point $z=a$.

→ If $z=a$ is pole of order 'n'.

$$\boxed{\left[\text{Residue of } f(z)\right]_{at\ z=a} = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]}$$

Sol

$\underline{\underline{Ex-1}}$ Find the Residue of $f(z) = \frac{e^{2z}}{(z-1)^3}$ at $z=1$.

Ans: $z=1$ is a pole of order 3

$$\therefore [Res]_{z=1} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^{m} \cdot f(z) \right].$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} \left[\cancel{(z-1)^3} \cdot \frac{e^{2z}}{\cancel{(z-1)^3}} \right]$$

$$= \frac{1}{2!} \times \lim_{z \to 1} 4e^{2z}$$

$$= 2e^2$$

$\underline{\underline{Ex-2}}$ Find the Residue of $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles.

Ans: $z = 0, 1, 2$ are the poles of order 1.

$$\rightarrow [Res]_{z=0} = \lim_{z \to 0} \cancel{(z-0)} \cdot \frac{1-2z}{\cancel{z}(z-1)(z-2)}$$

$$= \frac{1}{(-1)(-2)} = \frac{1}{2}.$$

$$\rightarrow [Res]_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \times \frac{1-2z}{z\cancel{(z-1)}(z-2)}$$

$$= 1.$$

$$\rightarrow [Res]_{z=2} = \lim_{z \to 2} \cancel{(z-2)} \cdot \frac{1-2z}{z(z-1)\cancel{(z-2)}}$$

$$= \frac{-3}{2(1)}$$

$$= -3/2$$

$\underline{\underline{Ex-3}}$ Find Residue of $f(z) = \frac{\quad}{(z+2)^2\,(z-2)^2}$ at $z = 2$.

$\underline{Ans}$: $z = 2$ is a pole of order 2.

$$\therefore [Res]_{z=2} = \lim_{z \to 2} \frac{d}{dz}\left[\cancel{(z-2)^2} \times \frac{1}{(z+2)^2 \cdot \cancel{(z-2)^2}}\right]$$

$$= \lim_{z \to 2} -\frac{2}{(z+2)^3}$$

$$= +\frac{1}{4} \cdot -\frac{2}{64}$$

$$= -\frac{1}{32}.$$

$\underline{\underline{Ex-4}}$ Find the Residue of $f(z) = \frac{1}{(z^2+1)^2}$ at $z = i$.

$\underline{\underline{Ans}}$: $z = i$ is a plole of order 2.

$$\therefore [Res]_{z=i} = \lim_{z \to i} \frac{d}{dz}\left[\cancel{(z-i)^2} \times \frac{1}{(z+i)^2\cancel{(z-i)^2}}\right]$$

$$= \lim_{z \to i} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(2i)^3}$$

$$= \frac{2}{8i}$$

$$= \frac{1}{4i}$$

$$\boxed{[Res]_{z=i} = -\frac{i}{4}}$$

$\dfrac{\quad}{z^3}$ at $z=0$.

Ans:

$$f(z) = \frac{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right]}{z^3}$$

$$= \frac{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \dots}{z^3}$$

$$f(z) = \frac{1}{6} - \frac{z^2}{5!} + \frac{z^5}{7!} + \dots$$

So, co-efficient of $\frac{1}{(z-a)}$ is zero.

So, $[\text{Res}]_{z=0} = 0.$

($\because$ $z=0$ is Removable singularity.)

Ex-5 Find Residue of $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}$ at $z=0$.

Ans:

$z=0$ is Pole of order 1.

So, $[\text{Res}]_{z=0} = \lim\limits_{z \to 0} (z-0) \cdot \dfrac{1+e^z}{(\sin z + z\cos z)}$

$$= \lim_{z \to 0} \frac{1 + e^z(z+1)}{\cos z + z\sin z + \cos z}$$

$$= \frac{1 + 1(0+1)}{1+0+1}$$

$$= \frac{2}{2}.$$

$[\text{Res}]_{z=0} = 1$

Ex-7 Find the Residue of $f(z) = e^{\frac{1}{z-a}}$ at $z=a$.

Ans: $z=a$ is a pole of order 1.

$$f(z) = e^{\frac{1}{z-a}}$$
$$= 1 + \frac{1}{(z-a)} + \frac{1}{2!(z-a)^2} + \frac{1}{3!(z-a)^3} + \cdots$$

Res. of $e^{\frac{1}{z-a}}$ at $z=a$ is coefficient of $\frac{1}{z-a}$ in the laurents expansion of fn f(z)

$$= e^{\frac{1}{z-a}}.$$

$$[Res]_{z=a} = 1.$$

Ex-8 $f(z) = e^{\frac{z}{z-2}}$. then find the Residue at $z=2$.

Ans:
$$e^{\frac{z}{z-2}} = 1 + \frac{z}{(z-2)} + \frac{z^2}{2!(z-2)^2} + \frac{z^3}{3!(z-2)^3} + \cdots$$

$$\rightarrow e^{\frac{z}{z-2}} = e^{\frac{z-2+2}{z-2}} = e^1 \cdot e^{\frac{2}{z-2}}.$$

$$= e\left[1 + \frac{2}{(z-2)} + \frac{(2)^2}{2!(z-2)^2} + \frac{(2)^3}{3!(z-2)^3} + \cdots\right].$$

So, $[Res]_{z=2} =$ Co-eff. of $\frac{1}{z-2}$

$$= 2e.$$

Ex- Find $z=-2.$

Ans: let, $z+2=u.$

$\therefore z=u-2.$

$\rightarrow (z-3)\sin\left(\frac{1}{z+2}\right) = (u-5)\cdot\sin\left(\frac{1}{u}\right).$

$= (u-5)\left[\frac{1}{u} - \frac{1}{3!\,u^3} + \cdots\right].$

$= 1 - \frac{5}{u} - \frac{1}{3!\,u^2} + \frac{5}{6u^3} + \cdots$

So, $[Res]_{u=0} = [Res]_{z=2} = $ co-eff. $\frac{1}{u}.$

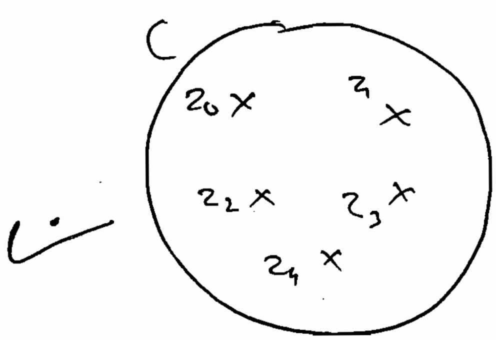
$[Res]_{z=2} = -5.$

☆ Cauchy's Residue Theorem:

→ Let, $f(z)$ is an analytic function within an a close curve c except at an finite nos. points then

$$\boxed{\int_c f(z)\,dz = 2\pi i\left[\text{sum. of Res. of } f(z) \text{ at its poles}\right].}$$

which lies within and on the curve c.



Ex- 1 $\int_C \frac{1+e^{z}}{\sin z + z\cos z}\, dz$ where

$z=0$ is a pole of order 1

Ans: $\int_C \frac{1+e^{z}}{\sin z + z\cos z}\, dz$

$= 2\pi i\, [\text{Res at } z=0]$.

$= 2\pi i\,(1)$.

$= 2\pi i$

Ex- 2 $\int_C \frac{z^2}{(z-1)^2 (z+2)}\, dz$ C is $|z-1| = 4$.

Ans: $z=1$ is poles of order 2 and

$z=-2$ is a pole of order 1. $\in |z-1| = 4$.

$\therefore f(z) = \frac{z^2}{(z-1)^2 (z+2)}$.

$\therefore [\text{Res}]_{z=1} = \lim_{z\to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{z^2}{\cancel{(z-1)^2} \cdot (z+2)}\right]$

$= \lim_{z\to 1} \frac{(z+2)(2z) - z^2}{(z+2)^2}$

$= \lim_{z\to 1} \frac{z^2 + 4z}{(z+2)^2}$.

$= \frac{1+4}{9}$

$[\text{Res}]_{z=1} = 5/9$.

Now, $[Res]_{z=-2} = \lim_{z \to -2} \ldots \frac{\ldots}{(z-1)^2 \cdot \cancel{(z+2)}}$

$$= \frac{4}{9}$$

$\therefore$ So, $\int_C \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i \left[R_1 + R_{-2} \right].$

$$= 2\pi i \left[\frac{5}{9} + \frac{4}{9} \right].$$

$$\int \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i$$

Ex-3 Find $\int_C \frac{e^z}{(z^2+1)} dz$ where 'c' is $|z| = 2$.

Ans: $z = \pm i$ is a pole of order 1 $\in |z| = 2$.

$$\therefore I = \int_C \frac{e^z}{(z^2+1)} dz = 2\pi i \left[R_i + R_{-i} \right].$$

$$\therefore [Res]_{z=i} = \lim_{z \to i} \frac{e^z \cdot \cancel{(z-i)}}{\cancel{(z-i)}(z+i)}$$

$$= \frac{e^i}{2i}.$$

$$[Res]_{z=-i} = \lim_{z \to -i} \frac{e^z \cdot \cancel{(z+i)}}{\cancel{(z+i)}(z-i)}$$

$$= \frac{e^{-i}}{-2i}$$

$$\therefore I = 2\pi i \left[\frac{e^i}{2i} - \frac{e^{-i}}{2i} \right].$$

$$I = \frac{2\pi}{2}\left[e^{i} - e^{-i}\right].$$

$$I = \pi\left[e^{i} - e^{-i}\right]$$

$$\therefore \quad I = 2i\frac{\pi}{\cancel{2}}\ \sinh i.$$

$$\therefore \quad I = 2\pi i \sinh i.$$

Ex - 4 $\int_C \tan z \, dz$ where 'c' is $|z| = 2$.

Ans: $f(z) = \tan z = \dfrac{\sin z}{\cos z}$.

$$\cos z = 0$$

$$\therefore z = (2n+1)\frac{\pi}{2}.$$

but $z = \pm \frac{\pi}{2}$ are lies inside $|z| = 2$.

$$\text{So,} \quad I = \int_C \tan z \, dz = 2\pi i \left[R_{\frac{\pi}{2}} + R_{-\frac{\pi}{2}} \right].$$

$$= 2\pi i \left[\lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \cdot \frac{\sin z}{\cos z} + \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \frac{\sin z}{\cos z} \right].$$

$$= 2\pi i \left[\lim_{z \to \frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)\cos z + \sin z}{-\sin z} + \lim_{z \to -\frac{\pi}{2}} \frac{\left(z + \frac{\pi}{2}\right)\cos z + \sin z}{-\sin\left(-\frac{\pi}{2}\right)} \right]$$

$$= 2\pi i \left[-1 - 1 \right].$$

$$= -4\pi i$$

Ans: $f(z) = e^{\frac{1}{z}}$

$$\therefore f(z) = 1 + \frac{1}{z} + \frac{1}{2!(z)^2} + \frac{1}{3!(z)^3} + \dots$$

$z=0$ is pole of order - 1.

$$\therefore [\text{Res}]_{z=0} = 1.$$

$$\therefore I = \int_C e^{\frac{1}{z}} . dz = 2\pi i \, [\text{ Res. at } z=0].$$

$$= 2\pi i.$$

* Evaluation of the Integrals of the form

$$\rightarrow \boxed{\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} . dx}$$

$\frac{f(x)}{F(x)}$ is a rational f^n of x satisfying

two condtions

(i) The degree of $F(x)$ is minimum 2 units more than the degree of $f(x)$.

(ii) $F(x) \neq 0$. for any real no.

then, $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} \cdot dx = 2\pi i$ [sum of the Res. of the f^n at its pole which lies in the Upper half Plane].

Ex-1 $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$,

Ans: $\frac{f(z)}{F(z)} = \frac{1}{1+z^2}$.

$\therefore z = \pm i$ are poles of order-1.

But $z = +i$ lies in upper half z-plane.

So, $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i$ [Res at $z = i$].

$= 2\pi i \left[\lim_{z \to i} \frac{\cancel{(z-i)}}{\cancel{(z-i)}(z+i)} \right]$

$= \frac{2\pi i}{2i}$

$= \pi$.

Ex-2 Expand $f(z) = \frac{1}{(z-1)\cdot(z-2)}$ in the region

① $|z|<1$ ② $1<|z|<2$. ③ $|z|>2$.

Ans: $f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$.

(i) $\because |z|<1$.

So, $z-2 = -2\left(1-\frac{z}{2}\right)$

$z-1 = -1(1-z)$.

$\therefore f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{(1-z)}$.

$$\therefore\ f(z) = \frac{}{2}\left[\qquad 2 \qquad 4 \qquad 8\right.$$

$$+ \left[1 + z + z^2 + z^3 + \cdots\right].$$

$$f(z) = 1 + \frac{3}{4}z + \frac{7}{8}z^2 + \cdots$$

(ii) $1 < |z| < 2.$

$$\therefore\ f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}.$$

$$= \cancel{\frac{1}{z(1+\frac{2}{z})}}\ \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}.$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right]$$

$$- \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right].$$

$$z \quad - \frac{1}{z} \quad \frac{1}{z}$$

$$f(z) = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} + \cdots - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \cdots$$

(iii) $|z| > 2.$

$$\therefore\ f(z) = \frac{1}{z\left(1 - 2/z\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}.$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right].$$

$$= 0 + \frac{1}{z} + \frac{3}{z^2} + \frac{7}{z^3} + \cdots$$

$$=$$

(i) if

i.e. only ~~terms~~ which ~~are~~ inverse terms.

e.g. $|z|>2$

$\Rightarrow \frac{2}{|z|}<1.$

(ii) if $|z|<$ secu. no. then take points inside

i.e only +ve power terms.

∴ e.g. $|z|<1.$

Ex-1 Expand $\frac{z}{(z+1)(z+2)}$ at $z=-2$.

Ans: let, $z+2=u.$

$\Rightarrow z=u-2.$

$$\therefore \frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)(u)}.$$

$$= \frac{(u-1)-1}{(u-1)u}.$$

$$= \frac{1}{u} - \frac{1}{u(u-1)}.$$

$$= \frac{1}{u} - \frac{1}{u-1} + \frac{1}{u}.$$

$$= \frac{2}{u} - \frac{1}{u-1}.$$

$$= \frac{2}{u} + [1+u+u^2+u^3+\cdots].$$

$$= \frac{2}{u} + 1 + u + u^2 + u^3 + \cdots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots$$

D: 30|5|2013

→ Basics
→ Probability
→ R-V | Expection
→ Dis^n → Discrete / → Continuous
→ Corre | Regression
mathmetical model.

→ Statistics:

→ it is a Process
1) collection of data
2) Analysis of data
3) Interpretation of data.

→ Befor applying formula, we have to check the type of data.

data → grouped / → ungrouped | Raw data

data → open / → close.

★ grouped data:

→ If data in form of Class Interval & freq, then it is called grouped data.

open data	closed data ✓
0 - 10	0 - 9
10 - 20	10 - 19
⋮	20 - 29
80 - 90	⋮
	79 - 80.

★ Ungrouped data.

→ It is based on observation.

Defination: According to Prof. R.A. Fisher "Statistics is defined as collection of data, Analysis of data and interpretation of data.

* Types of Datas:

- Grouped & Ungrouped data.
- Open & Close.

☆ Defination of Grouped data:

→ If the data in the form of a class Interval and freq. together, then the data is known as grouped data. OR distributing the freq. to their corresponding intervals is known as freq. distribution.

☆ Closed data:

→ If the class intervals are in continous form without any discontinuity then the data is known as closed data. otherwise data is known as open data.

☆ Ungrouped data:

→ If the data contains only observations without any class intervals then the data is known as ungrouped data. OR Raw data.

→ For ungrouped data:

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

x_i → Individual observation

n → no of. observation

→ For Grouped data:

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N}}$$

x_i : mid point $= \frac{UL + LL}{2}$

N : Sum of freq. | Total.

☆ **Median: (ungrouped data)**

→ If n is odd then middle observation itself is median.

→ If n is even then avg. of middle obser. is median. provided:

(i) the data is rearranged in assending or dessending order.

(ii) no. of observation above the middle is equal to the no. of observation below the middle.

(*) **Median for Grouped data:**

$$\boxed{M_d = L + \left(\frac{N/2 - m}{f}\right) c.}$$

→ where L = Lower Limit of ideal class

f = freq. for ideal class.

→ m = Cummulative freq. for [illegible] class.

c = class interval.

Ex. Find the median of following freq. data.

→

C-I	freq.	m
0 – 10	3	3
10 – 20	5	8 → m
20 – 30	7	15
30 – 40	2	17
40 – 50	1	18
	N = 18	

Ideal class

$\frac{N}{2} = \frac{18}{2} = 9.$

$\therefore L = 20 \qquad m = 15.$

$f = 7$

$$M_d = L + \left(\frac{\frac{N}{2} - m}{f}\right) c$$

$c = 30 - 20$

$c = 10.$

$$M_d = 20 + \left(\frac{9 - 8}{7}\right)(10)$$

$$M_d = 20 + \frac{10}{7} = 21.4$$

$\therefore \boxed{M_d = 21.4 \text{ (approx).}}$

if $\frac{N}{2} = 17$ then 30 - 40 is ideal.

if $\frac{N}{2} = 3$ then 0 - 10

Note! whenever the First class itself is ideal then Commulitive freq. ~~is~~ and freq. are equal.

$\therefore \boxed{f = m}$ ✓

The most frequently repeated observation is known as mode:

E.g.: 1, 2, 3, 4, 5, 2, 8, 7, 2, 3, 11, 14, 21, 43, 3, 51.

$M_o = 2$ (unimodal)

$M_o = 2, 3$ (Bimodal).

★ Mode :- (Grouped data).

$$\boxed{M_o = L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) c}$$

Where, $\Delta_1 = f - f_{-1}$.

$\Delta_2 = f - f_{+1}$.

E.g. find the mode for Grouped data.

→

C.I.	freq.
0 - 2	11
2 - 4	14
4 - 6	17
6 - 8	8
8 - 10	3

$f_{-1} = 14$

$f_{+1} = 8$.

Ideal class

* Ideal class is one which has highest freq.

$$M_o = L + \left(\frac{\Delta 1}{\Delta 1 + \Delta 2}\right) c$$

$\Delta 1 = f - f_{-1}$.

$= 17 - 14 = 3$

$\Delta 2 = f - f_{+1}$

$= 17 - 8 = 9$.

$$= 4 + \left(\frac{3}{3+9}\right) 2.$$

$$= 4 + 6/12$$

$$\therefore \boxed{M_o = 4.5}$$

$\boxed{\text{Note:}}$ (1) Maximum freq.s are repeated first (?) & Last in between. then select the in between ideal. (Unimodal).

(2) If the maximum freq are repeated in betn select randomly. (Bimodal).

(3) If all the freq. are equal
Mode is undefined. ($\frac{0}{0}$ form).

(4) If the maximum freq.s are repeated 1st & Last select the randomly.

☆ Measures of Central Tendencies.:

→ Mean (Best).
Medium.
mode.

☆ Measures of Dispersion | variabilites:

- Range.
- Quartile Deviation. (QD)
- Mean Deviation. (MD).
- Coefficient of ~~Deviation~~. Variation. (CV)
- Standard Deviation. (SD).

* Range:

$$\begin{bmatrix} \text{Max} - \text{min} \\ \text{G.V.} - \text{L.V.} \end{bmatrix}$$

* Variance: (Ungrouped data)

→ Taking the deviation or differences of data from its mean is known as variance.

$$\boxed{\begin{aligned} \sqrt{\text{Variance}} &= S.D. \\ \text{Variance} &= S.D.^2 \end{aligned}}$$

$\therefore$

$$\boxed{\sigma_x^2 = \frac{\sum (x_i - \bar{x})^2}{n}}$$

$$\boxed{\sigma_x^2 = \frac{1}{n} \sum x_i^2 - (\bar{x})^2}$$

Point: (1) Lesser variances is more consistance or more uniform.

(2) Variance never be negative.

(3) Variance of constant is zero.

(4) Variance of the variable is positive.

(5) Sum of the deviation from its mean is always zero.

(6) Sum of the squares of the deviation from its mean should be minimum.

* Variance (Grouped data):

$$\boxed{\sigma_x^2 = \frac{1}{N} \sum f_i x_i^2 - (\bar{x})^2}$$

$$\boxed{\sigma_x^2 = \frac{1}{N} \sum f_i (x_i - \bar{x})^2}$$

☆ Relation betⁿ QD | MD | SD.

$$\boxed{6\,QD = 5\,MD = 4\,SD.}$$

∴ $6\,QD = 4\,SD.$

∴ $QD = \frac{4}{6}\,SD.$

∴ $\boxed{QD = \frac{2}{3}\,SD.}$

∴ $\boxed{QD = \frac{2}{3}\,\sigma.}$

$5\,MD = 4\,SD.$

∴ $\boxed{MD = \frac{4}{5}\,SD}$

* Coefficient of Variation:

∴ $C.V. = \frac{S.D.}{Mean} \times 100.$

∴ $\boxed{C.V. = \frac{\sigma}{\bar{x}} \times 100.}$

✓

∴ **Note:** (1) Lesser σ implies lesser C.V. that implies data is more consistance or More uniform.

(2) For identifying the Consistancy within the data it can be measurable with Standard deviation as well as coefficient of Variation.

$\rightarrow$

S_1	n_1	$\bar{x}_1$	σ_1^2
S_2	n_2	$\bar{x}_2$	σ_2^2

$$\boxed{\text{Comb } \bar{X} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}}$$

$$\text{Comb } \sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1+n_2}$$

where, $d_1 = \bar{x}_1 - \bar{x}$.

$d_2 = \bar{x}_2 - \bar{x}$.

Ex. : Find the mean and variance of first n natural No.

$\rightarrow$ $1, 2, 3, \ldots, n$.

$$\bar{x} = \frac{1+2+3+\cdots+n}{n} = \frac{1}{n}[1+2+3+\cdots+n].$$

$$\therefore \bar{x} = \frac{n(n+1)}{\cancel{n}\,2n}.$$

$$\therefore \boxed{\bar{x} = \frac{n+1}{2}}$$

$$\therefore \sigma_x^2 = \frac{1}{n}\Sigma x_i^2 - (\bar{x})^2.$$

$$\rightarrow \frac{1}{n}\Sigma x_i^2 = \frac{1}{n}[1^2+2^2+\cdots+n^2]$$

$$= \frac{1}{\cancel{n}} \times \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}.$$

Now, $$\sigma_x^2 = \frac{1}{n}\Sigma x_i^2 - (\bar{x})^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \left(\frac{n+1}{2}\right)\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \left(\frac{n+1}{2}\right)\left[\frac{4n+2-3n-3}{6}\right].$$

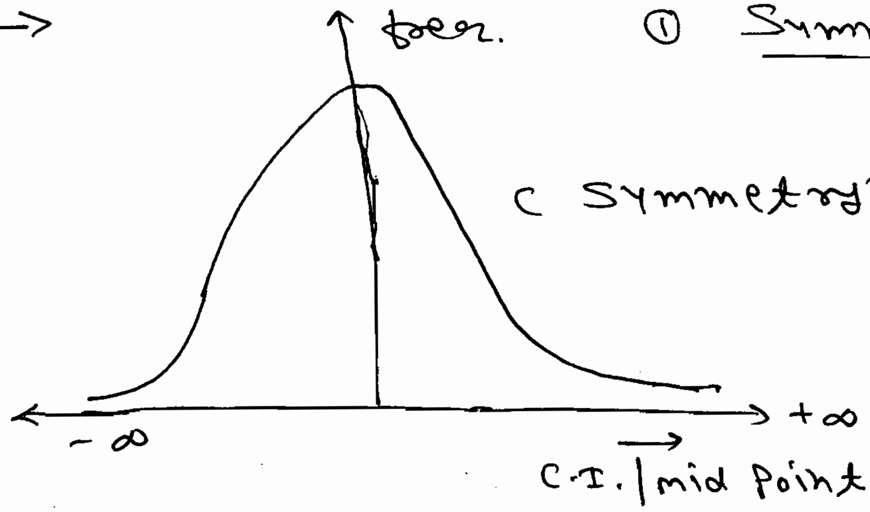
$$= \left(\frac{n+1}{2}\right)\left[\frac{n-1}{6}\right]$$

$$\therefore \boxed{\sigma_x^2 = \frac{n^2-1}{12}}$$

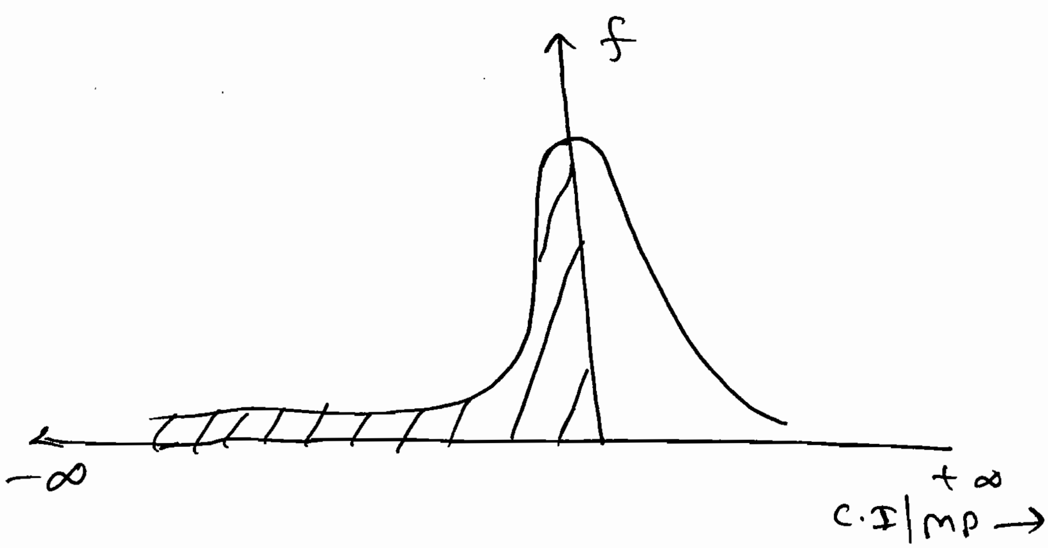
☆ Skewness:

→ It is a geometrical representation of the freq. curve and is defined lack of symmetry.

→ ① Symmetry

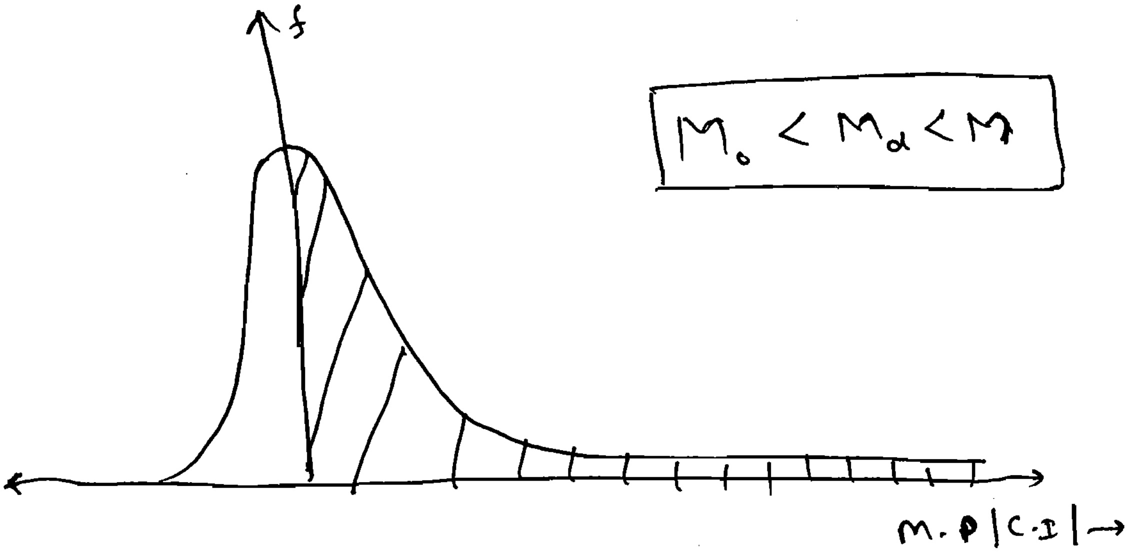


→ ② Skewed -ve:



✓ $\boxed{M_o > M_d > M.}$

(3) +ve Skewed:



$$\boxed{M_o < M_d < M}$$

* Pearson's coefficient of Skewness:

$$\boxed{S_{kp} = \frac{M - M_o}{\sigma}}$$

$$\boxed{M_o = 3M_d - 2M.}$$

$$\therefore \quad S_{kp} = \frac{3(M - M_d)}{\sigma}.$$

$$\boxed{\therefore \; -3 < S_{kp} < +3.}$$ ← Emperical Measurement.

(1) Random Experiment:

→ Unpredictible outcomes of an experiment is known as random experiment.

e.g. - Tossing a unbiased coin.

- Rolling a dice.
- Drawing a card from a deck of 52.

(2) Sample Space:

→ The Collection of all possible outcomes of an experiment is known as Sample Space.

→ It is denoted by S.

(3) Event:

→ The outcomes of an experiment is known as event.

→ Mathematically event is a Subset of the Sample Space.

(4) Probability:

→ The Probability of an event is defined as the ratio of favourable cases to the event to the no. of outcomes of an expeperiment. (The outcomes are mutually exclusive, joint exhaustive event)

$$\boxed{P(E) = \frac{m}{n}}, \quad \text{where, } m \leq n.$$

* Axiometic Approach | Probability function:

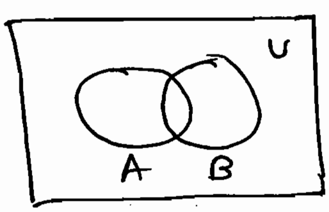
→ $P(S) = 1$

→ $0 < P(E) \leq 1$

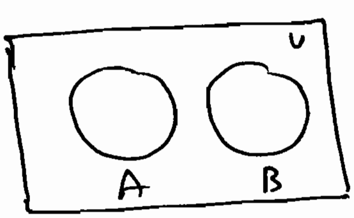
- $P(E) = 0$ → Impossible.
- $P(\emptyset) = 0$ → Empty. | Null
- $P(E) = 1$ → certain | sure.

→ $$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

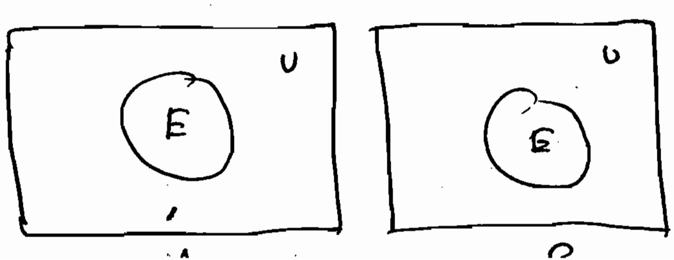
where, E_i must be disjoint or mutually exclusive event.

⇒ 

→ Dependent event.

⇒ 

→ Mutually exclusive event

⇒ 

→ Independent

Points:

(1) Occurrence of one event doesn't depends upon the other occurrence of ~~other~~ events on the Same Sample Space, then those events are known as mutually exclusive events.

(2) Let, A & B are mutually exclusive events

$$\boxed{\begin{gathered} A \cap B = \phi \\ P(A \cap B) = 0. \end{gathered}}$$

and

(3) Occurrence of one event doesn't depend upon the occurrence of a same event in a different Sample space then those events are known as independent events.

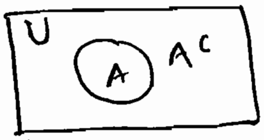
(4) Mutually exclusive events never be equal to independent events and independent events never be mutually exclusive events.

Results:

→ $P(S) = 1$

→ $0 < P(E) \leq 1$.

→ $P(A^c) = 1 - P(A)$.



$A \cup A^c = S$

$P(A \cup A^c) = 1$.

$\therefore$ $P(A) + P(A^c) = 1$.

$\therefore$
- $P(A^c) = 1 - P(A)$.
- $P(A) = 1 - P(A^c)$.

→ This known as <u>Complementry theorem</u>.

(1) <u>Addition theorem for dependent events:</u>

→ If A & B two events

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$$

(2) <u>Addition theorem for mutually exclusive events.</u>

→ If A & B are Mutually exclusive events.

then

- $P(A \cup B) = P(A) + P(B)$ $\quad (\because P(A \cap B) = 0)$
- $P(A + B) = P(A) + P(B)$.
- $P(A + B + C) = P(A) + P(B) + P(C)$.

Dependent event:

→ If A & B are two events.

$$\boxed{P(A\cap B) = P(A)\cdot P(B/A) = P(B)\cdot P(A|B)}$$

Conditional probability

known

unknown

→ If A, B, C are three events.

$$\boxed{P(A\cap B\cap C) = \underbrace{P(A)\cdot P(B|A)}_{P(A\cap B)}\cdot P(C|A\cap B)}$$

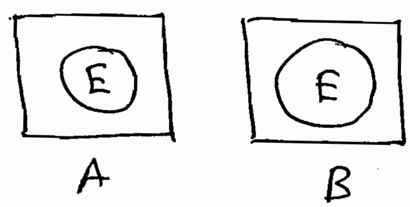
(4) Multiplication theorem for Independent:

→ If A & B are two independent events.

$$P(A\cap B) = P(A)\cdot P(B).$$

$(\because P(B/A) = P(B))$.

$(\because P(A|B) = P(A))$.



→ If $E_1, E_2, E_3, \dots E_n$ are I.E.

$$P(E_1\cap E_2\cap E_3\cap \dots E_n) = P(E_1)\cdot P(E_2)\cdot P(E_3)\dots P(E_n).$$

→ $P(A\cap B) = P(A)\cdot P(B|A) = P(B)\cdot P(A|B).$

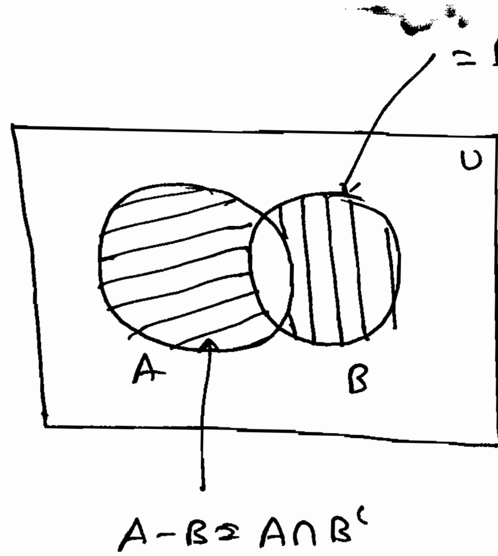
$\therefore$ (1) $P(A|B) = \dfrac{P(A)}{P(B)} \times P(B|A).$

(2) $P(B|A) = \dfrac{P(B)}{P(A)} \times P(A|B).$

→ If A & B are two events.

⊙ only A occured

$$\boxed{P(A \cap B^c) = P(A) - P(A \cap B).}$$



⊙ only B occured.

$$\boxed{P(A^c \cap B) = P(B) - P(A \cap B).}$$

⊙ neither A nor B.

$$\boxed{P(A^c \cap B^c) = P(\overline{A \cup B}) = 1 - P(A \cup B).}$$

⊙ only once.

$A \Delta B = (A - B) \cup (B - A).$

$$\boxed{P(A \Delta B) = P(A \cap B^c) + P(A^c \cap B).}$$

$\Rightarrow$ $P(A^c | B) = \dfrac{P(A^c \cap B)}{P(B)}.$

$$= \frac{P(B) - P(A \cap B)}{P(B)}.$$

$$= 1 - \frac{P(A \cap B)}{P(B)}.$$

$\therefore P(A^c | B) = 1 - P(A|B).$

$\Rightarrow$ $P(A | B^c) = \dfrac{P(A \cap B^c)}{P(B^c)}.$

$P(A|B^c) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}.$ $(\because P(B) \neq 1)$

$$P(B^c)$$

$$P(A^c|B^c) = \frac{1 - P(A\cup B)}{1 - P(B)} \quad (\because P(B) \neq 1).$$

* For Mutually exclusive and Exhustic Events

→ If A & B are two mutually exclusive and exhustic events. the

$$P(A\cap B) = 0 \quad (\because A\cap B = \emptyset).$$

$$P(A\cup B) = P(A) + P(B) = 1.$$

→ $P(A\cap B^c) = P(A) - P(A\cap B) = P(A).$

$\therefore \boxed{P(A\cap B^c) = P(A).}$

→ $P(A^c\cap B) = P(B) - P(A\cap B) = P(B).$

$\therefore \boxed{P(A^c\cap B) = P(B).}$

→ $P(A^c\cap B^c) = 1 - P(A\cup B) = 1 - (P(A)+P(B))$
$= 1 - 1 = 0.$

$\therefore \boxed{P(A^c\cap B^c) = 0.}$

→ $P(A^c|B) = \frac{P(A^c\cap B)}{P(B)} = \frac{P(B) - P(A\cap B)}{P(B)}.$

$$= \frac{P(B) - 0}{P(B)} = 1.$$

$\therefore \boxed{P(A^c|B) = 1.}$

→ $P(A^c|B) = \frac{P(A^c\cap B)}{P(B)} = \frac{P(B) - P(A\cap B)}{P(B)} = 1.$

→ $P(A|B^c) = P(A)/1-P(B) = \frac{P(A)}{P(A)} = 1.$

$1 - P(A\cup B)$

Note:

(1) If A & B are independent events, $P(A \cap B^c)$, $P(A^c \cap B)$ & $P(A^c \cap B^c)$ are also independent.

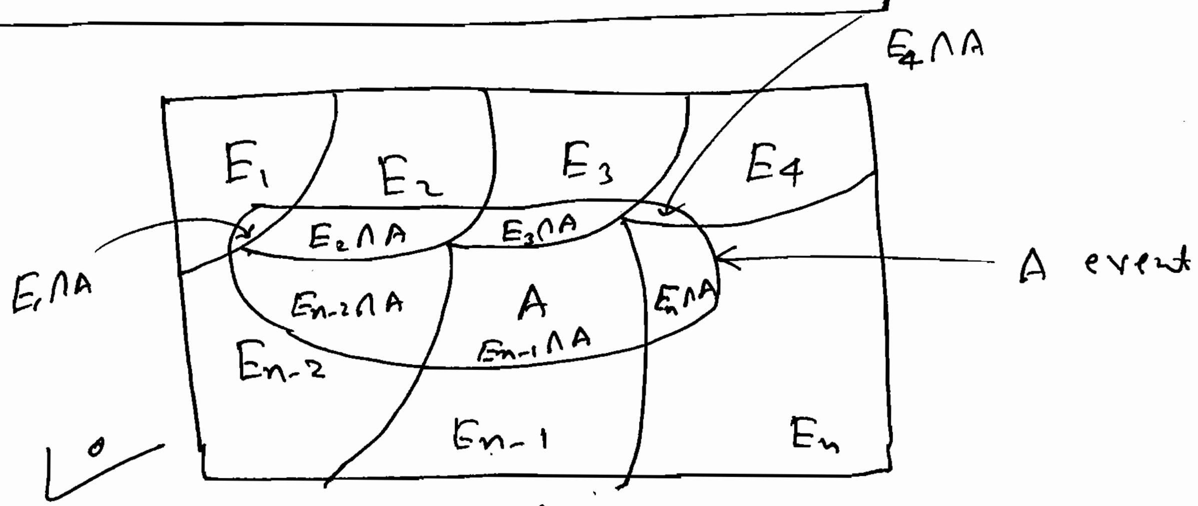
☆ Baye's Theorem:

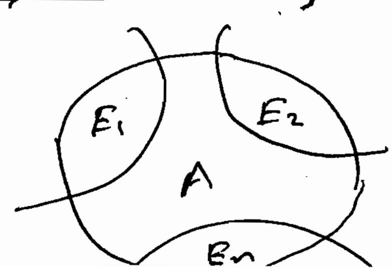
→ If $E_1, E_2, \ldots E_n$ are mutually exclusive events ($P(E_i) \neq 0$) such that A arbitrary event which is subset of "$\bigcup_{i=1}^{n} E_i$" then P

$$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + \cdots + P(E_n \cap A).$$

$$\therefore P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \cdots + P(E_n) \cdot P(A|E_n).$$

$$\therefore \boxed{P(A) = \sum_{i=0}^{n} P(E_i) \cdot P(A|E_i).}$$ → Total Prob. Known event.





$$P(E_i/A) = \frac{\quad}{P(A)}.$$

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\cdot P(A|E_i)}.$$

⇓

Reverse prob. known.

* Steps in the Baye's theorem:

- Identify the known events in the data. (Mutually exclusive). $P(E_1)$, $P(E_2)$, ...
- Select the unknown events. i.e. event A (Part of the known events).
- Write a prob. of unknown in terms of known. $P(A|E_1)$, $P(A|E_2)$, ...
- Find the total prob. of unknown events. i.e $P(A)$
- Compute Reverse prob. for known events. $P(E_i/A)$

Complete Information for [illegible]

Problems:

⇒ 1- coin → 2

2- coin → 2^2 (2 → no of coins; 2 ↓ no. of occurrence.)

⋮

n- coin → 2^n.

⇒ 1- dice → 6

2- dice → 6^2

⋮

n- dice → 6^n.

⇒ 52 cards

		K	Q	J
	13 - Hearts	1 } R 26	1	1
	13 - diamond	1	1	1
	13 - clubes	1 } B 26	1	1
	13 - ~~Suads~~ spreds	1	1	1
	52	4	4	4 = 12

= 12 no. of face cards or picture cards.

			Addition Th.	MUL. The.
⇒ at least	min	≥	→ either / or	- simultaneously
at most	max	≤	→ at least once	- one after other
and	Product	∩	→ or	- as well as
OR	Sum	∪		- successively
				- alternatively
				- one by one
				- and

find the prob. to getting atmost one head.

Ans. $n(S) = 2^3 = 8$. Let x = head.

H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

$\therefore P(X \leq 1) = P(X < 1) + P(X = 1)$.

$= P(X = 0) + P(X = 1)$.

$= 1/8 + 3/8$

$= 4/8$

$= 1/2$.

Cont.

Ex. Find ~~th~~ prob. that at least one tail.

$\therefore P(X \geq 1) = 1 - P(X < 1)$.

X = no. of tails.

$\therefore P(X \geq 1) = 1 - P(X = 0)$.

$= 1 - 1/8$

$\therefore P(X \geq 1) = 7/8$.

Ex. Find the prob. that at least one head and atmost one tail.

Ans:

	1st coin	2nd coin	3rd coin
1 T	T	H	H
	H	T	H
	H	H	T
or			
0 T	H	H	H

$P(A) = \frac{4}{8}$

$= \frac{1}{2}$.

Ex. Find the prob. of atmost one [illegible] at most one tail.

Ans: $p = 0/8 = 0$.

($\because$ non of the outcomes contains one H & one T).

Ex-2 Four coins are tossed at a time find the prob. getting at least / at most two heads and at least / atmost two tails.

Ans: $n(S) = 2^4 = 16$.

P (at least two heads and at least two tails)

$= 6/16 = 3/8$.

HHTT
TTHH
HTTH
THHT
HTHT
THTH

~~= 6 possible ways. to arr~~

$= 6. = {}^4C_2$

4 → repeats

2 → favourable.

$${}^4C_0^H = 1 = {}^4C_0^T$$

$${}^4C_1^H = 4 = {}^4C_3^T$$

$${}^4C_2^H = 6 = {}^4C_2^T$$

$${}^4C_3^H = 4 = {}^4C_1^T$$

$${}^4C_0^H = \frac{1}{16} = {}^4C_4^T.$$

Ex: Find a prob that [illegible]

∴ at most 2 T = $6/16 = 3/8$

Ex. Find a prob. of no. of H = no. of T

∴ Ans: $6/16 = 3/8$

Ex-3 A coin is reapet a 6 times Find the prob. that no. of Heads are more than the no. of tails.

Ans: P (no. of H's > no. of T's) =

$$= \frac{{}^6C_4 + {}^6C_5 + {}^6C_6}{64}$$

$$= \frac{15+6+1}{64}$$

$$= 22/64$$

H

${}^6C_0$ ×

${}^6C_1$ ×

${}^6C_2$ ×

${}^6C_3$ ×

${}^6C_4$ ✓

${}^6C_5$ ✓

${}^6C_6$ ✓

✡ ✡ ✡ Ex-4 A coin is repeated n- times find the prob. the head appears in the odd no. of times. $n(S) = 2^n$.

Ans: All ~~odd~~ even bianomical coeff. = All odd bianomical coeff.

∴ $C_0 + C_2 + C_4 + \dots + C_n = C_1 + C_3 + \dots + C_{n-1}$.

Req. prob $= \frac{2^{n-1}}{2^n} = 1/2$.

Ex-5 Two dice are rolled ... Find the prob. that first two dice contain a prime no. or a total of 8.

Ans. $n(S) = 6^2 = 36$.

A : P No. first Dice.

B : total 8 $P(B) = 5/36$.

8 : (5, 3), (3, 5)
(2, 6), (6, 2)
(4, 4)

2	3	5
(2,1)	(3,1)	(5,1)
(2,3)	⋮	⋮
(2,4)		
(2,2)		
(2,5)		
(2,6)	(3,6)	(5,6)
6	6	6

$P(A) = 6$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{18}{36} + \frac{5}{36} - 3/36$

$= 20/36$.

Ex-6 2 dice are rolled two times. Find the prob. that for getting a sum of 7.
(1) at least once
(2) only once.
(3) twice.

Ans: This is independent event.

(1) at least once:

P(at least

A : Sum 7 'f.t'.

B : Sum 7 's.t'.

$P(B) = 6/36 = 1/6 \Rightarrow P(B^c) = 5/6.$

① $P($ ~~only~~ at least once$) := P(A \cup B)$

$$= P(A) + P(B) - P(A) \cdot P(B).$$
$$= 1/6 + 1/6 - 1/6 \times 1/6.$$
$$= \cancel{2/6} - 1/36$$
$$= 11/36.$$

② $P($ only once$) := P(A \cap B^c) + P(A' \cap B).$

$$= P(A) \cdot P(B^c) + P(A^c) \cdot P(B).$$
$$= 1/6 \times 5/6 + 5/6 \times 1/6.$$
$$= 10/36.$$

③ $P($ twice$) = P(A \cap B)$

$$= P(A) \cdot P(B).$$
$$= 1/6 \times 1/6$$
$$= 1/36.$$

Ex-6: 2 dice are rolled. Find the prob. that neither sum 9 nor sum 12.

12	11	10	9	8	7
6	5	4	3	2	1

Ans: $P(\text{sum } 9) = 4/36.$

$P(\text{sum } 12) = 1/36.$

ME

$$P(9^c \cap 12^c) = 1 - P(9 \cup 12)$$
$$= 1 - [P(9) + P(12)].$$
$$= 1 - [4/36 + 1/36]$$

Ex-7 4 cards are drawn from 52 cards. Find the prob. that

① All 4 cards from the same suit.

② ~~No.~~ No 2 cards are drawn from the same suit.

Ans:

③

$$\begin{array}{l} \\ P(\text{4 cards same suit}) = \\ \text{(at a time)} \end{array} \frac{\overset{H}{{}^{13}c_4} + \overset{D}{{}^{13}c_4} + \overset{C}{{}^{13}c_4} + \overset{S}{{}^{13}c_4}}{{}^{52}c_4}.$$

$\therefore$ P(No 2 cards on same suit) (one by one) $= \dfrac{{}^{13}c_1 + {}^{13}c_1 + {}^{13}c_1 + {}^{13}c_1}{{}^{52}c_4}$.

Ex-8 A determinant is choosen from the set of all determinants of order 2 with the elements 0 and (or) 1. Find the prob. that the choosen determinant is non-zero.

Ans: $n(S) = 16$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\Delta = ad - bc$.

Case-(i): $\Delta = +1$ [$a = d = 1$ at least one of $b = c = 0$]

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 3.$$

Case-(ii) $\Delta = -1$. [$b = c = 1$ at least one of $a = d = 0$].

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 3.$$

$P(\text{non neg } \Delta) = 1 - 3/16 = 13/16.$

$P(\text{zero } \Delta) = 1 - 6/16 = 10/16.$

Ex-1 A & B are the two players rolling a dice on the condition that one who ~~constant~~ gets the two first winning the game. If A starts the game what are the winning chances of player A, Player B.

Ans: $P(e) = 1/6$, $P(e^c) = 5/6$.
'p' 'q'

$$\rightarrow P(\text{win } B) = \underset{\text{trial-1}}{qp} + \underset{\text{trial-2}}{q^2 qp} + \underset{\text{trial-3}}{q^2 . q^2 . qp} + q^2 \cdot q^2 q^2 qp + \cdots$$

Losser winner

$$= qp \left[1 + q^2 + q^4 + q^6 + \cdots \right].$$

$$= qp \left[\frac{1}{1-q^2} \right] \qquad \left(\because a_n = \frac{a}{1-r} \right).$$

$$= \frac{5}{6} \times \frac{1}{6} \left[\frac{1}{1 - \frac{25}{36}} \right]$$

$$\therefore \boxed{P(\text{win } B) = 5/11.}$$

$$\rightarrow P(\text{win } A) = p + pq^2 + q^2 . q^2 p + \cdots$$

$$= p \left[1 + q^2 + q^4 + \cdots \right]$$

$$= p \left[\frac{1}{1-q^2} \right]$$

$$= \left(\frac{1}{6}\right) \left[\frac{1}{1-(5/6)^2} \right]$$

$$= \frac{1}{6} \times \frac{364}{9}$$

$\therefore$ P(win A) = 2/...

A, B, C ... a coin in the same order. On the condition that one who gets the Heads first winning the game. If A starts the game what are the winning chances of for player C in third trial.

Ans: $P(H) = 1/2$, $P(T) = 1/2$.

'p' Success   'q' Failer.

$\rightarrow P(\text{win } C) = qqp \rightarrow 1^{st}$ trial ×

$= q^3 qqp \rightarrow 2^{nd}$ trial. ×

$\boxed{= q^3 \cdot q^3 \cdot qqp \rightarrow 3^{rd} \text{ trial.}}$

$\therefore P(\text{win } c) = q^8 \cdot p = (1/2)^8 \cdot (1/2) = \frac{1}{512}$.

$\therefore \boxed{P(\text{win } c) = \frac{1}{512}}$

Ex 3 A dice rolled, if the no. is the even no. Find the prob. composite no.

Ans: $S = \{ 1, 2, 3, 4, 5, 6 \}$

C = unknown.

E = known.

$\therefore P(C/E) = \frac{P(C \cap E)}{P(E)}$

$= \frac{2/6}{3/6}$

$= 2/3$

Ex-3 A card is drawn from 52 card a card is red card find the prob. that it is diamond.

Ans: $P(D|R) = \dfrac{P(D \cap R)}{P(R)}$

$= \dfrac{13/52}{26/52}$

$= \dfrac{1}{2}$.

Ex-4 A no. is choosen from the 100 no.s those are 00, 01, 02, ..., 99. Let x denotes the sum of digits on a no. and y denotes the product of the digits of the no. find the prob. that $P(x=9 | y=0)$.

Ans: $P(x=9 | y=0) = \dfrac{P(x \cap y)}{P(y)}$

$= \dfrac{2/100}{19/100}$

$= \dfrac{2}{19}$.

Ex-5 60 % Employees from the Company are College graduates. out of this 10 % in the sales department. the Employees who didn't graduate from the college are 80 % in the sales the department. A person is selected at random. find the prob. that

① The person in the sales dep.

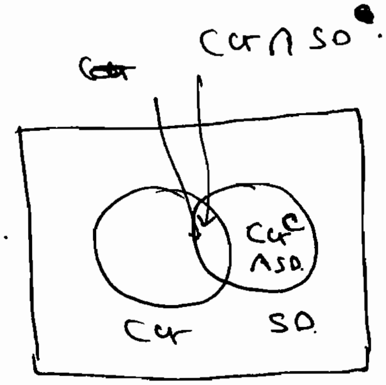
② The person neither in the sales dep

$P(Ca^c) = 40\% = 0.4.$

$\rightarrow$ $P(SO)/Z$

$P(SO|Ca) = 10\% = 0.1.$

$\therefore P(SO|Ca^c) = 0.8.$



$$\therefore P(SO) = P(Ca \cap SO) + P(Ca^c \cap SO).$$

$$= P(Ca) \cdot P\left(\frac{SO}{Ca}\right) + P(Ca^c) \cdot P\left(\frac{SO}{Ca^c}\right).$$

$$= (0.6)(0.1) + (0.8) \cdot (0.4).$$

$$P(SO) = 0.38.$$

② $P(Ca^c \cap SO^c) = 1 - P(Ca \cup SO).$

$$= 1 - [P(Ca) + P(SO) - P(Ca \cap SO)]$$

$$= 1 - \left[P(Ca) + P(SO) - P(Ca) \cdot P\left(\frac{SO}{Ca}\right)\right]$$

$$= 1 - [0.6 + 0.38 - 0.6 \times 0.1].$$

Ex-6 to Ex-9 are from Baye's theorem

$$= 0.08.$$

Ex-6 There are three coins, out of this 2 are unbiased and one is biased with two heads. A coin is drawn at random and tossed two times. It appears head on both the times. Find the prob. that it is from the biased coin.

Ans: $2 \quad 1 \quad = 3.$

Step-1

$\rightarrow P(UB) = \dfrac{{}^{2}C_{1}}{{}^{3}C_{1}} = 2/3.$

$\therefore P(B) = \dfrac{{}^{1}C_{1}}{{}^{1}C_{3}} = 1/3.$

Step-2

E : Getting a head 2 times.

$P(E/UB) = 1/2 \times 1/2 = 1/4.$

$P(E|B) = 1 \times 1 = 1.$

Step-3 $\quad P(E) = P(UB \cap E) + P(B \cap E).$

$= P(UB) \cdot P(E|UB) + P(B) \cdot P(E|B).$

$= 2/3 \times 1/4 + 1/3 \times 1.$

$= \dfrac{2}{12} + 1/3$

$= 1/2.$

Step-4 $\quad P(B|E) = \dfrac{P(E \cap B)}{P(E)} = \dfrac{P(B) \cdot P(E/B)}{P(E)}.$

$= \dfrac{1/3 \times 1}{1/2} = 2/3.$

$P(UB|E) = \dfrac{2/3 \times 1/4}{1/2} = 1/3.$

he select the no. from 1 to 5. If the tail appears he selects the no. from 1 to 10. Find ① the prob. that the selected no. is a even no.

② If the even no. is happed what is the prob. for getting head.

Ans:

→ $P(H) = 1/2$.

$P(T) = 1/2$.

→ ∴ E = Getting Even no.

∴ $P(E|H) = 2/5$.

$P(E|T) = 5/10 = 1/2$.

→ $P(E) = P(H \cap E) + P(T \cap E)$.

$= P(H) \cdot P(E|H) + P(T) \cdot P(E|T)$.

$= (1/2) \times (2/5) + (1/2) \cdot (1/2)$.

∴ $P(E) = 9/20$.

→ $P(H/E) = \dfrac{P(H \cap E)}{P(E)}$.

$= \dfrac{P(H) \cdot P(E|H)}{P(E)}$.

$= \dfrac{1/2 \times 2/5}{9/20}$

$\boxed{P(H|E) = 4/9}$.

∴ $\boxed{P(T/E) = 5/9.}$

knows the answer or guess the answer.
Let, P the Prob. that Student knowing ~~an~~ ans to a que. and 1 - P be the prob. that Student Guessing ~~the~~ the ans. to a que. Assume that if the student gets the ans to a que will be correct with prob. 1/5. What is the conditional prob. that if the student knew the ans. to a que given that he answered correctly.

Ans: $\rightarrow P(K) = P \quad : \quad P(G) = 1 - P.$

$\rightarrow$ E : ~~getting~~ answering correctly.

$\rightarrow \quad P(E|K) = 1.$

$P(E|G) = 1/5.$

$\rightarrow \quad P(E) = P(K \cap E) + P(G \cap E).$

$= P(K) \cdot P(E|K) + P(G) \cdot P(E|G).$

$= P \cdot 1 + (1-P)\, 1/5.$

$P(E) = \frac{4P-1}{5}.$

$\rightarrow \quad P(K|E) = \frac{P(K \cap E)}{P(E)}.$

$= \frac{P(K) \cdot P(E|K)}{P(E)}.$

$= \frac{P \cdot 1}{\frac{4P+1}{5}} = \frac{5P}{4P+1}$ //

Contain Blue, Red and Green colour of ~~the~~ balls in the form of ~~1, 2, 3~~.

	B	R	G	
A	1	2	3	= 6
B	2	3	1	= 6
C	3	1	2	= 6

Bags — Must remember

A bag is drawn at random and two balls are taken from it. They are found to be one Blue & one Red. Find the prob. that the choosen balls are from bag C.

Ans: → $P(A) = 1/3$.

$P(B) = 1/3$.

$P(C) = 1/3$.

→ E: Getting a 1 R & 1 B.

$$P(E/A) = \frac{{}^{2}C_{1} * {}^{1}C_{1}}{{}^{6}C_{2}}$$

${}^{2}C_{1}$ → No. of ways to get 1 Red ball from 2 Red balls.

${}^{1}C_{1}$ → No. of ways to get 1 Blue ball from 1 Blue ball.

${}^{6}C_{2}$ → No. of ways to get 2 balls from bag A

$\therefore P(E/A) = 2/15$.

$$\therefore P(E/B) = \frac{{}^{2}C_{1} \times {}^{3}C_{1}}{{}^{6}C_{2}} = 6/15.$$

$$\therefore P(E/C) = \frac{{}^{3}C_{1} \times {}^{1}C_{1}}{{}^{6}C_{2}} = 3/15.$$

$\therefore \cancel{P(E|}$

$\rightarrow P(E) = P(A \cap E) + P(B \cap E) + P(C \cap E).$

$$= P(A) \cdot P(E|A) + P(B) \cdot P(E|B) + P(C) \cdot P(E|C).$$

$$= \frac{1}{3}\left[\frac{2}{15} + \frac{6}{15} + \frac{3}{15}\right] = \frac{11}{45}.$$

$$\rightarrow \therefore P(C|E) = \frac{P(C \cap E)}{P(E)} = \frac{1/3 \times 3/15}{11/45} = 3/11.$$

$$\therefore P(B|E) = \frac{P(B \cap E)}{P(E)} = \frac{P(B) \cdot P(E|B)}{P(E)}$$

$$= \frac{1/3 \times \frac{6}{15}}{11/45} = 6/11.$$

$$\therefore P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A) \cdot P(E|A)}{P(E)}.$$

$$= \frac{1/3 \times 2/15}{11/45} = \frac{2}{11}.$$

<u>Gate</u> <u>Exam:</u>

① $\{1, 2, 3, 4, \ldots 6\}.$

* Random Variables & Expectation

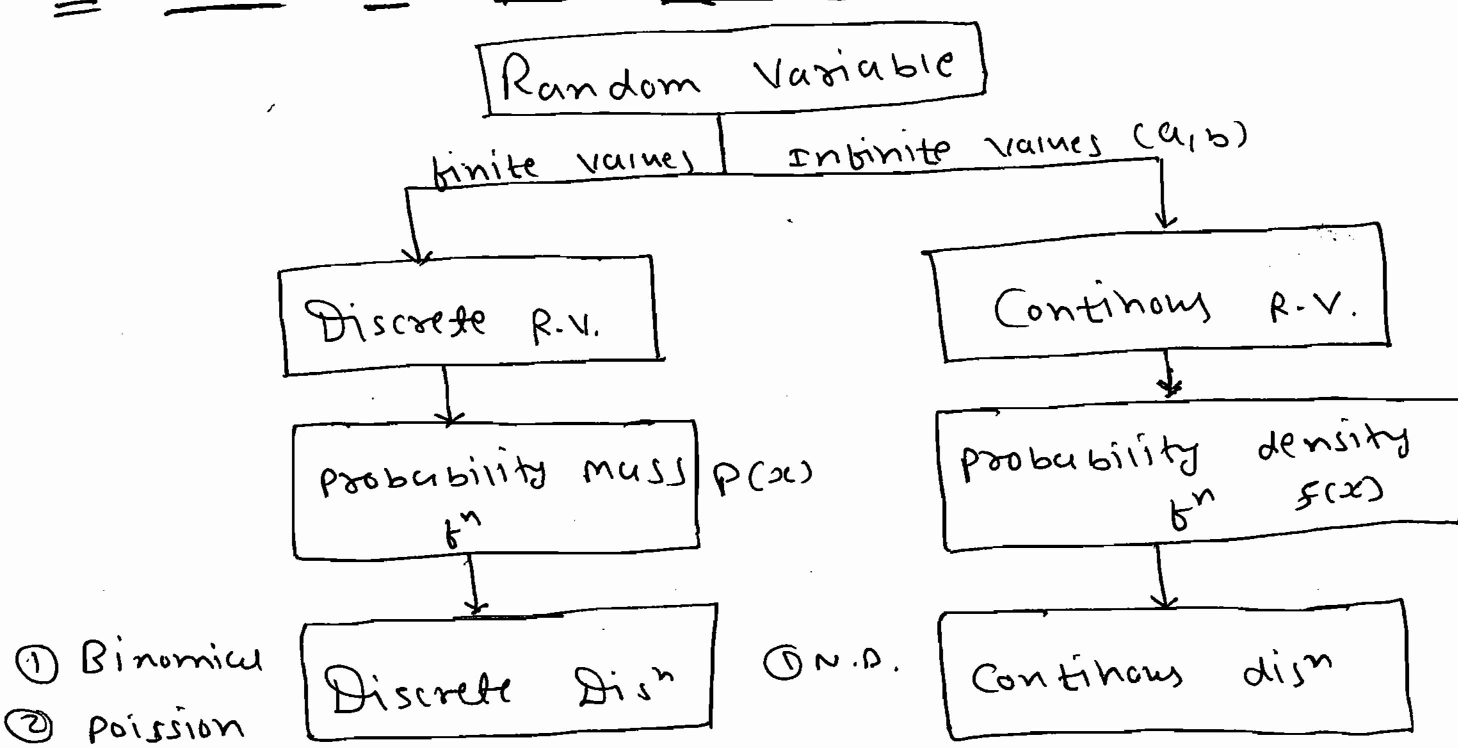
* Random Variable:

⇒ Connecting the outcomes of an experiment with a real values is known as random variable. (1D Random Variable). The corresponding data is known as univariate data.

* 2D Random Variable:

⇒ Connecting the 2 outcomes of an experiment at a time 2 real values, provided that those 2 outcomes drawn from same sample space. The corresponding data is known as Bivariate data.

* Types of the Random Variable:



Note:
$$\frac{d F(x)}{dx} = f(x).$$

$$\therefore \boxed{F(x) = \int_{-\infty}^{x} f(x) . dx.}$$

* Expection (Mean):

$$\rightarrow \boxed{\bar{X} = E[X] = \sum_{i=1}^{n} x_i \cdot P(x_i)}$$

where, X = Descrete R.V.

$$\rightarrow \boxed{\bar{X} = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$

where, X = Cont. R.V.

$$\left(\because \sum_{i=1}^{n} p(x) = 1 \quad \int_{-\infty}^{\infty} f(x) = 1 \right)$$

* Variance:

$$V(x) = E(x^2) - (E(x))^2.$$

$$= E(x - E(x))^2$$

$$V(x) = \sum x^2 . p(x) - \left(\sum x . p(x) \right)^2 \text{ for } \boxed{X = DRV}$$

$$\boxed{V(X) = \overline{X^2} - (\bar{X})^2}$$

$$\therefore V(x) = \int x^2 . f(x) - \left(\int x . f(x) \right)^2 . \text{ for } \boxed{X = CRV,}$$

Note:
Variance: Mean square - Square mean.

→ If X & Y are two R.V.

$\therefore$ $\boxed{E(X+Y) = E(X) + E(Y).}$

$\therefore$ $\boxed{E(X-Y) = E(X) - E(Y).}$

→ If X & Y are 2 R.V.,

$$\boxed{E(aX) = a\,E(X).}$$

→ If X & Y are 2 R.V.,

$$\therefore\ \boxed{\left.\begin{aligned} E(X\cdot Y) &= E(X)\cdot E(Y|X). \\ &= E(Y)\cdot E(X|Y). \end{aligned}\right\} \rightarrow \text{conditional expectation.}}$$

→ If X & Y Independent r.v.'s

$$\boxed{E(X\cdot Y) = E(X)\cdot E(Y).}$$

* If $Y = aX + b$: a, b constant.

$$\begin{aligned} E(Y) &= E(aX+b) \\ &= E(aX) + E(b). \\ &= a\,E(X) + b. \end{aligned}$$

* $E($~~const~~ constant$)$ = constant

$$\boxed{E(E(E(X))) = E(X).}$$

* Properties of Variance:

→ If X & Y Independent R.V.'s.

$$\boxed{V(X+Y) = V(X) + V(Y)}$$

$= V(X) + V(-Y)$.

$= V(X) + V(Y)$.

$\therefore \boxed{V(X \pm Y) = V(X) + V(Y)}$.

→ If X is r.v. and a is constant.

$\boxed{V(aX) = a^2 V(X)}$.

$\therefore V(-Y) = (-1)^2 V(Y). = V(Y)$.

$\therefore \boxed{V(-Y) = V(Y)}$.

→ If X, Y are independent R.V. & a & b const

$$\boxed{V(aX + bY) = a^2 V(X) + b^2 V(Y)}$$

$$\boxed{V(aX - bY) = a^2 V(X) + b^2 V(Y)}$$

$$\boxed{V\left(X/a \pm Y/b\right) = \frac{V(X)}{a^2} + \frac{1}{b^2} V(Y)}$$

→ If $Y = aX + b$: a, b constant

$\therefore V(aX+b) = V(aX) + V(b)$.

$= a^2 V(X) + 0$ $\left(\because \boxed{V(\text{const.}) = 0}\right)$.

→ If X & Y are R.V.

$$\therefore \boxed{V(X+Y) = V(X) + V(Y) + 2\,\mathrm{cov}(X, Y)}$$

$$\boxed{V(X-Y) = V(X) + V(Y) - 2\,\mathrm{cov}(X, Y)}$$

→ $\mathrm{cov}(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$.

If a, b const.

$$\boxed{\mathrm{cov}(a, b) = E(a \cdot b) - E(a) \cdot E(b)}$$

→ If X & Y are independent R.V. then

$\boxed{cov(X,Y) = 0.}$

✓ but Converse of the statement is not true.

→ ~~Mean and Variance are independe~~

→ Mean is dependent of change of origin & also dependent of scale.

→ Variance and co-variance are independent of change of origin as well as dependent of change of scales.

* Skewness:

→ It is lack of symmetry.

$$\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}.$$

$\mu_3 =$ 3rd central moment

$\mu_2 =$ variance.

Note: → If a $\mu_3 = 0 \Rightarrow \beta_1 = 0$. then the curve symmetry.

→ If μ_3 is −ve then the curve is −vely skewed.

→ If μ_3 is +ve then the curve is +vely skewed.

Ex-1 Find the Expection of the no. of dice when it is thrown.

Ans:

$x_{r.v.}$	1	2	3	4	5	6
$p(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\text{Mean} = E(X) = \sum_{i=1}^{n} p(x) \cdot x.$$

$$= 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4p(4) + 5p(5) + 6p(6).$$

✓

$$= 1/6 \left[1 + 2 + 3 + 4 + 5 + 6\right]$$

$$= 21/6$$

$$\therefore \boxed{E(X) = 7/2 \text{ or } 3 \text{ or } 4.}$$

→ Mean of dice is $7/2$.

Ex-2 Find the Variance on the dice.

Ans: $V(x) = E(X^2) - (E(X))^2.$

$$\rightarrow E(X^2) = \sum_{1}^{6} x^2 p(X).$$

$$= 1^2 p(1) + 2^2 p(2) + 3^2 p(3) + 4^2 p(4) + 5^2 p(5) + 6^2 p(6).$$

$$= 1/6 \left[1 + 4 + 9 + 16 + 25 + 36\right].$$

$$E(x^2) = 91/6.$$

$$\therefore V(x) = 91/6 - (7/2)^2$$

$$= \frac{91}{6} - \frac{49}{4} = \frac{182 - 147}{12} = \frac{35}{12} \approx 3.$$

Note: The Mean and variance for the sum of the no. on the dice is,

$$\boxed{E(x) = \frac{7n}{2}}$$

$$\boxed{V(x) = \frac{35n}{12}}$$

where, n is no. of dice.

Ex-2 Three unbiased dice are thrown find the Mean and variance for the sum of the nos on them.

Ans:

$n = 3$.

$$E(x) = \frac{7n}{2} = \frac{7 \times 3}{2} = \frac{21}{2}.$$

$$V(x) = \frac{35}{12} \cdot n = \frac{35}{12} \times 3 = \frac{35}{4}.$$

Ex-3 Two unbiased dice are thrown. Find the $E(x)$ for the sum 7.

Ans:

$$E(x) = x \cdot P(x).$$

we have only one R.V. $\Rightarrow$ 7.

$$E(7) = 7 \cdot P(7).$$

$$= 7 \cdot (6/36).$$

$$E(7) = 7/6$$

Ex-4 A player tossed 3 coins, he win 500 Rs. if a 3 heads occured, 300 Rs. if 2 heads occured, 100 Rs. for only one head occured. on the other hand he losses 1500 Rs. if three tails occurred. Find value of the game.

Ans:

$$E(x) = 500 \times \left(\frac{1}{8}\right) + 300\left(\frac{3}{8}\right) + 100\left(3/8\right) - 1500\left(1/8\right).$$

200 = Rs 25/-

x no. of heads (R.V).	3	2	1	0
$P(X)$	$1/8$	$3/8$	$3/8$	$1/8$

$^3C_3$ $^3C_2$ $^3C_1$ $^3C_0$

$\therefore$ Value of game = Gain - Loss

$$= (500 \times 1/8 + 300 \times 3/8 + 100 \times 3/8) - (1500 \times 1/8).$$

$$= \frac{200}{8}$$

$$= 25/-.$$

$\boxed{\text{Note:}}$ if a gain is said to be the fail, the expected value of game is zero.
(No Loss & No gain).

Ex-6. A man has given 100 keys out of which one fit a lock. He tries them successively without replacement to open the lock. what is the probability that the lock will be open at the 49th trial. Also determine Mean and variance.

Ans: $\boxed{\text{Note:}}$ → with replacement ⇒ Independent trials.
→ without replacement ⇒ Dependent events.

→ ① without replacement :

Prob. of drawing key in 1st trial = $1/100$

prob. of " " " 2nd trial = $1/99$

3rd trial = $1/98$

$$2^{nd} \text{ trial} = \left(1-\frac{1}{100}\right) \times \frac{1}{99}$$

$$= \frac{99}{100} \times \frac{1}{99} = \frac{1}{100}.$$

$$3^{rd} \text{ trial} = \left(1-\frac{1}{100}\right) \times \left(1-\frac{1}{99}\right) \times \frac{1}{98} = \frac{1}{100}.$$

$$= \frac{\cancel{99}}{100} \times \frac{\cancel{98}}{\cancel{99}} \times \frac{1}{\cancel{98}} = \frac{1}{100}.$$

$\therefore$ prob. of opening the lock 1^{st} success in 49^{th} trial $= \frac{1}{100}$.

Now, $n = 100$

$$\text{mean} = E(x) = \frac{n+1}{2} = \frac{100+1}{2} = 50.5.$$

$$V(x) = \frac{n^2-1}{12} = \frac{(100)^2-1}{12}.$$

(2) <u>with replacement.</u>

$$P(49^{th} \text{ trial}) = \left(1-\frac{1}{100}\right)^{48} \cdot \frac{1}{100}.$$

$$= \frac{(99)^{48}}{(100)^{49}}.$$

$\boxed{\text{Notes:}}$ → The proba. for 1^{st} success in the r^{th} trial by <u>with replacement</u> technique is

$$\therefore \boxed{P(x = r^{th} \text{ trial}) = q^{r-1} \cdot p.}$$

where, q is a failure prob.

p is a success prob.

→ The prob. of the 1^{st} success in the r^{th} trial <u>without replacement</u> is $\boxed{1/n.}$ where n is no. of given observation.

Ex - 8 If a x is ... fn is $f(x) = k \cdot x^2$. x ranges from 0-1

Ans. a-1 Find the value of k.

a-2 mean and variance.

$\Rightarrow$ (i) since, $\int_0^1 f(x) = 1.$

$\therefore \int_0^1 k \cdot x^2 = 1.$

$\therefore k \left[\frac{x^3}{3}\right]_0^1 = 1.$

$\therefore \boxed{k = 3.}$

(ii) Mean

$E(x) = \int_0^1 x \cdot f(x) \cdot dx.$

$= \int_0^1 x \cdot 3x^2 \cdot dx.$

$= 3 \times \left[\frac{x^4}{4}\right]_0^1$

$\boxed{E(x) = 3/4.}$

(iii) variance:

$\therefore V(x) = E(x^2) - (E(x))^2$

$E(x^2) = \int_0^1 x^2 \cdot f(x) \cdot dx$

$= \int_0^1 x^2 \cdot 3x^2 \cdot dx.$

$= 3/5.$

$\therefore \boxed{V(x) = 3/80}$

$\therefore V(x) = 3/5 - 9/16 = \frac{48-45}{80} = 3/80$

find the $V(X)$.

Ans: Mean $= E(X) = \int_{-1}^{1} x \cdot f(x) \cdot dx$.

$$= \int_{-1}^{1} \underbrace{x}_{\text{odd}} \cdot \underbrace{|x|}_{\text{Even}} \cdot dx.$$

$$= 0. \qquad \left(\because \int_{-1}^{1} f(x) = 0, \; f(x) = \text{odd}. \quad \int_{-1}^{1} f(x) \cdot dx = 2 \int_{0}^{1} f(x), \; f(x) = \text{even} \right).$$

$\therefore$ Variance $V(X) = E(X^2) - (E(X))^2$

$$= \int_{-1}^{1} x^2 \cdot |x| \cdot dx - 0$$

$$= 2 \int_{0}^{1} x^3 \cdot dx.$$

$$= 2 \times \left[\frac{x^4}{4} \right]_0^1$$

$$\therefore \boxed{V(X) = \tfrac{1}{2}.}$$

Ex-10 if ~~x~~ x is R.V. and $f(x) = K \cdot x^2 \cdot e^{-x}$. $0 \leq x \leq \infty$. Find (1) K

(2) $E(X)$ & $V(X)$.

Ans: **Note:** Gamma: f^n:

$$\boxed{\Gamma n = \int_{0}^{\infty} x^{n-1} \cdot e^{-x} \cdot dx, \quad n > 0.}$$

$\therefore$ $\Gamma n = (n-1)!$

$\Gamma 1 = 1.$

Since, $\int_0^{\infty} f(x) = 1.$

$$\therefore \int_0^{\infty} x^2 . e^{-x} . dx = 1.$$

$$\therefore k \sqrt{n} = 1.$$

$$\therefore \cancel{k\sqrt{}} \quad k \sqrt{3} = 1.$$

$$\therefore k (n-1)! = 1$$

$$\therefore k (2!) = 1.$$

$$\boxed{k = 1/2}$$

$$\therefore E(x) = \int_0^{\infty} x . f(x) . dx.$$

$$= \int_0^{\infty} \frac{x^3}{2} . e^{-x} . dx =$$

$$= \sqrt{4} . /2$$

$$= (n-1)!/2$$

$$= 3!./2$$

$$\boxed{E(x) = 3}$$

$$\therefore E(x^2) = \frac{1}{2}\int_0^{\infty} x^4 . e^{-x} . dx.$$

$$= \sqrt{5} = 4! = 24/2 = 12.$$

$$\therefore V(x) = E(x^2) - (E(x))^2$$

$$= 12 - 9$$

$$\therefore \boxed{V(x) = 3}$$

$E(X) = 10$, $V(X) = 25$. find the value a, b such that $y = ax - b$ has expectation is zero and $V(y) = 1$.

Ans: $E(X) = 10$, $V(X) = 25$.

$E(y) = 0$.

$\therefore E(ax - b) = 0$.

$\therefore aE(X) - b = 0$.

$\therefore \boxed{10a - b = 0}$

$\therefore b = 10a$

$\boxed{b = \pm 2.}$

$V(y) = 1$.

$\therefore V(ax - b) = 1$.

$\therefore a^2 V(X) + V(b) = 1$.

$\therefore a^2 V(X) = 1$.

$\therefore 25a^2 = 1$

$\boxed{a = \pm 1/5}$

$\therefore$ but had asked the value

so $\boxed{a = 1/5}$, $\boxed{b = 2.}$

* Bia Variant Data:

$\boxed{\text{Case - I}}$: Continuous Random Variable.

→ if x & y are 2D CRV and its PDF is known as joint PDF. is denoted by $f(x, y)$.

→ The MDF are

→ $$\boxed{f(x) = \int_y f(x, y)\, dy.}$$

→ $$\boxed{f(y) = \int_x f(x, y)\, dx.}$$

→ If x & y are 2D CRV, independenat CRV if and only if

$$\boxed{f(x, y) = f(x) \cdot f(y).}$$

$$\therefore \boxed{Jpdf = mdf(x) \cdot mdf(y).}$$

→ Relation betn JDf | Jpdf.

$$\boxed{\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y).}$$

or

$$\boxed{F(X, Y) = \int_{-\infty}^{X} \int_{-\infty}^{Y} f(x, y) \cdot dx\, dy.}$$

$$f(x|y) = \frac{f(x, y)}{f(y)}.$$

$$f(y|x) = \frac{f(x, y)}{f(x)}.$$

→ If X & Y are 2D DRV and it pdf is known as Joint prob. mass f^n. (JPMf) is denoted by $P(X, Y)$.

→ The Marginal mass functions (MMf) are

$$\boxed{\begin{array}{l} P(X) = \sum_{y} P(X, Y). \\ \hline P(Y) = \sum_{x} P(X, Y). \end{array}}$$

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Ex-1 If X & Y are 2D DRV and it JPDf is

X \ Y	-1	0	+1
-1	1/4		
0		1/2	
+1			1/4.

find $P(X+Y=2 \mid X-Y=0)$.

Ans: $P(X+Y=2 \mid X-Y=0) = \dfrac{P(X+Y=2 \cap X-Y=0)}{P(X-Y)=0}$

$$= \frac{P(X=1, Y=1)}{P(X=1, Y=1) + P(X=0, Y=0) + P(X=-1, Y=-1)}.$$

$$= \frac{1/4}{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}}$$

$$= \frac{1}{4}.$$

Binomial Distribution:

⇒ **Def^n:**

→ If x is said to be a binomial random variable allowes the value from 0 to n with the parameter n, p and its pmf is

$$\therefore \boxed{B(x, n, p) = P(x) = \binom{n}{x} p^{x} q^{n-x},}$$

$0 \leq x \leq n$

$p + q = 1.$

$q = 1 - p.$

$= 0,$ Otherwise.

* Conditions:

(i) Observations are independent (n is small)

(ii) Prob. of Success is constant (p is large).

(iii) Mean is greater than the Variance.

* Properties:

$E(X) =$ Mean $= np.$
$V(X) = \mu_2 = npq.$
$\mu_3 = npq(q-p). = npq(1-2p).$
$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$
$\therefore \beta_1 = \frac{(1-2p)^2}{}$

$\Rightarrow P = 1/2 \Rightarrow \mu_3 = 0$. then the curve symmetry.

$\Rightarrow p < 1/2 \Rightarrow \mu_3 = +ve$ then the curve is +very skewed.

$\Rightarrow P > 1/2 \Rightarrow \mu_3 = -ve$ then the curve is -very skewed.

$\rightarrow$ Sum of the independent binomial R.V.s is also a binomial R.V.

Ex-1 Find the Prob. of getting a sum 9 exactly 2 in 3 times with a pair of dice.

Ans:

$n = 3$

$x = 2$

p = getting 9 $= \frac{4}{36}$, ($\because$ (4,5); (5,4) (3,6), (6,3).)

$= 1/9$

$q = 1 - 1/9$

$q = 8/9$

$\therefore \quad P(x=2) = \left({}^{n}c_{x}\right) p^{x} \cdot q^{n-x}$

$= \left({}^{3}c_{2}\right) \cdot \left(\frac{1}{9}\right)^{2} \cdot \left(\frac{8}{9}\right)^{3-2}$

$= \cancel{3} \times \frac{1}{81} \times \frac{8}{\cancel{9}3}$

$\therefore \boxed{P(x=2) = \frac{8}{243}}$

Ex-2 Prob. of man hitting a target [cut off]

☆ ① If he fire the 5 times what is the Prob. of his hitting a target atleast twice.

② How many times must he fire so that the prob. his hitting the target at least once. ~~hi~~ is more than 90%.

Ans: $p = 1/3$, $q = 1 - 1/3 = 2/3$.

(i) $n = 5$.

$\therefore P(x \geq 2) = 1 - [P(x=0) + P(x=1)]$.

$$= 1 - \left[({}^5c_0)\, p^0 \cdot (q)^n + {}^5c_1 (1/3)^{1} \cdot (2/3)^4 \right]$$

$P(x \geq 2) = \frac{131}{243}$.

(ii) $n = 5$.

$P(x \geq 1) = 1 - P(X=0). > 90\%$

$\therefore 1 - ({}^nc_0)(1/3)^0 \cdot \left(\frac{2}{3}\right)^n > 0.9$.

$\therefore 1 - \left(\frac{2}{3}\right)^n > 0.9$

$\therefore \left(\frac{2}{3}\right)^n < 0.1$.

$\therefore n \log\left(\frac{2}{3}\right) < \log 0.1$.

$\therefore \boxed{n = 5.672 \approx 6.}$

* find avg. no. of times in which the no. on the 1^{st} dice is exceeds the no. on the 2^{nd} dice.

Ans: the no. on the 1^{st} dice > no. on the 2^{nd} dice.

$\therefore$ (2,1), (3,1), (3,2).

(4,1) (4,2), (4,3).

(5,1), (5,2), (5,3), (5,4).

(6,1), (6,2), (6,3), (6,4), (6,5)

= 15.

$\therefore \quad p = \frac{15}{36}.$

$\therefore$ Mean = $E(x) = n.p. = 120 \times \frac{15}{36}$

$= 50.$

*

Ex-4 X & Y are the Binomial R.V.s.

$X \sim$ (follows) $B(2,p)$

$Y \sim B(4,p)$.

if $P(X \geq 1) = 5/9$, find

(i) $P(Y \geq 1)$.

(ii) $P(X+Y \geq 1)$.

$n_x = 2$

$n_y = 4.$

Ans: $P(x \geq 1) = 5/9.$

$1 - P(X=0) = 5/9.$

$\therefore \quad 1 - q^{n_x} = 5/9.$

$\therefore \quad 1 - q^2 = 5/9.$

$\therefore \quad q^2 = 4/9$

$\therefore \quad q = 2/3.$

$\therefore \quad p = 1 - 2/3$

$p = 1/3$

Now, $P(y \geq 1) = 1 - ...$

$$= 1 - q^{n_y}$$

$$= 1 - (2/3)^4.$$

$$= 1 - 16/81$$

$$P(y \geq 1) = 65/81.$$

$$\therefore \quad P(x+y \geq 1) = 1 - P(x+y=0).$$

$$= 1 - q^{n_x + n_y}.$$

$$= 1 - (2/3)^{2+4}.$$

$$= 1 - \frac{64}{}$$

Ex-6 If x is a binomial R.V. then find the value of $\sum_{x=0}^{n} \left(\frac{x}{n}\right) \left({}^{n}c_{x}\right) p^{x} \cdot q^{(n-x)}$

Ans: $\sum_{x=0}^{n} \left(\frac{x}{n}\right) \left({}^{n}c_{x}\right) p^{x} \cdot q^{n-x}.$

$$= \frac{1}{n}\left[\sum_{x=0}^{n} x \cdot \left({}^{n}c_{x} \cdot p^{x} \cdot q^{n-x}\right)\right].$$

$$= \frac{1}{n}\left[\sum_{x=0}^{n} x \cdot P(x)\right].$$

$$= \frac{E(x)}{n}$$

$$= \frac{\cancel{n} \cdot p}{\cancel{n}}$$

$$= p.$$

⇒ Defn:

→ If x is said to be a point of R.V. define in the $[0, \infty]$ with parameter λ (>0) and its PMF is

$$\therefore \boxed{P(x, y>0) = P(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}}, \quad \lambda > 0, \ 0 \le x < \infty$$

$$= 0, \quad \text{otherwise.}$$

* Conditions:

→ Observations are infinity large.

→ Either Prob. is success very small.

→ $np = \lambda \Rightarrow p = \lambda / n$.

$$\boxed{P(x : n, p) = \frac{e^{-np} \cdot (np)^{x}}{x!}}$$

it is a approximation binomial.

→ $$\boxed{P(x : \lambda, t > 0) = \frac{e^{-\lambda t} \cdot (\lambda t)^{x}}{x!}}$$

it is a poission process.

** Properties:

→ $E(X) = \text{Mean} = \lambda$

→ $V(X) = \mu_2 = \lambda$.

→ $\mu_3 = \lambda$.

$\frac{}{x^3} = x$

NOTE:

(i) In Poisson dis^n Mean = variance = parameter $= \lambda$.

(ii) It is always +vely skewed. ($\because \lambda > 0 \Rightarrow \beta > 0$).

(iii) Sum of the Independent Poissons R.V. is also a poisson R.V.

(iv) Diff^n bet^n the independent poisson's R.V. is not a Poisson R.V.

Ex-1 A Telephone switch board receives 20 calls on an avg. during an hour. Find the prob. that for a period of 5 min.

(1) No call received.

(2) Exactly 3 calls are Received.

(3) At least 2 calls are received.

Ans:

	λ
$60'$	20.
$1'$	$\frac{20}{60} = \frac{1}{3}$.
$5'$	$\frac{1}{3} \times 5 = 5/3. = 1.65. = \lambda$.

(i) $P(X=0) = \dfrac{e^{-1.65} \cdot (1.65)^0}{0!} = e^{-1.65}$.

31.

(iii) $P(X \geq 2) = 1 - P(X<2)$.

$$= 1 - [P(X=0) + P(X=1)].$$

$$= 1 - \left[e^{-1.65} + \frac{(1.65) \cdot e^{-1.65}}{1!} \right].$$

Ex-2 If x_1 & x_2 are two independent poisson R.V. with variance $(1, 2)$. Find $P(X_1 + X_2 = 4)$.

Ans: $$P(X_1 + X_2 = k) = \frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^k}{k!}.$$

An ⇒ Here. variance is 1 & 2 for x_1 & x_2.

$\therefore$ $V(x_1) = 1 = \lambda_1$

$V(x_2) = 2. = \lambda_2$

$$\therefore P(x_1 + x_2 = 4) = \frac{e^{-(3)} \cdot (3)^4}{4!}.$$

Ex-3 If X & Y are two independent R.V. such that ~~pro~~ $P(X=1) = P(X=2)$ & $P(Y=2) = P(Y=3)$ find $V(3X - 4Y)$.

Ans: $P(X=1) = P(X=2)$.

$$\therefore \frac{\cancel{e^{-\lambda}} \cdot (\lambda)^1}{1!} = \frac{\cancel{e^{-\lambda}} \cdot (\lambda)^2}{2!}.$$

② $\theta > 0$.

$$\therefore \frac{\cancel{e^{-\theta}} \cdot (\theta)^2}{2!} = \frac{\cancel{e^{-\theta}} \cdot (\theta)^3}{3!}.$$

$\therefore \quad 3 = \theta.$

$\therefore \quad \theta = 3$

$\therefore \quad E(Y) = V(Y) = 3.$

Now, $\quad V(3X - 4Y)$

$= 9V(X) + 16V(Y).$

$= 9(2) + 16(3).$

$= 18 + 48.$

$V(3X - 4Y) = 66.$

Ex - 4 If X is Poisson r.v. & $E(X^2) = 6$ find $V(X)$.

Ans:

$V(X) = E(X^2) - (E(X))^2$

$V(X) = E(X) = \lambda.$

$\therefore \quad V(X) = 6 - (V(X))^2.$

$\therefore \quad \lambda^2 + \lambda - 6 = 0.$

$\therefore \quad \lambda = 2, \ \boxed{\lambda = -3.} \rightarrow$ not possible.

$\therefore \quad \boxed{\lambda = 2} \Rightarrow \boxed{V(X) = 2}$

find $\displaystyle\sum_{x=0}^{\infty} \frac{x}{\lambda} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

Ans:

$$\sum_{x=0}^{\infty} \frac{x}{\lambda} \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}.$$

$$= \frac{1}{\lambda} \left[\sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \right].$$

$$= \frac{1}{\lambda} \left[\sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \right].$$

$$= \frac{E(x)}{\lambda}.$$

$$= 1.$$

D: 4 | 6 | 2012

☆ Normal Disⁿ: [Gaussian]

⇒ Defⁿ:

→ If x is said to be a Normal R.V. define in the interval [$-\infty, +\infty$] with mean is equal to μ and variance = σ^2 then the R.V. is known as normal R.V. and its density fⁿ is

$$\therefore N(x : \mu, \sigma^2) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} .$$

$$-\infty < x < +\infty$$

$$= 0 \text{, otherwise} \quad -\infty < \mu < +\infty$$

$$0 < \sigma < \infty .$$

⇒ Standard normal R.V.

→ If x is a normal R.V. with mean = 0 and $\sigma^2 = 1$. then the R.V. is known as standard normal R.V. and its density fⁿ is

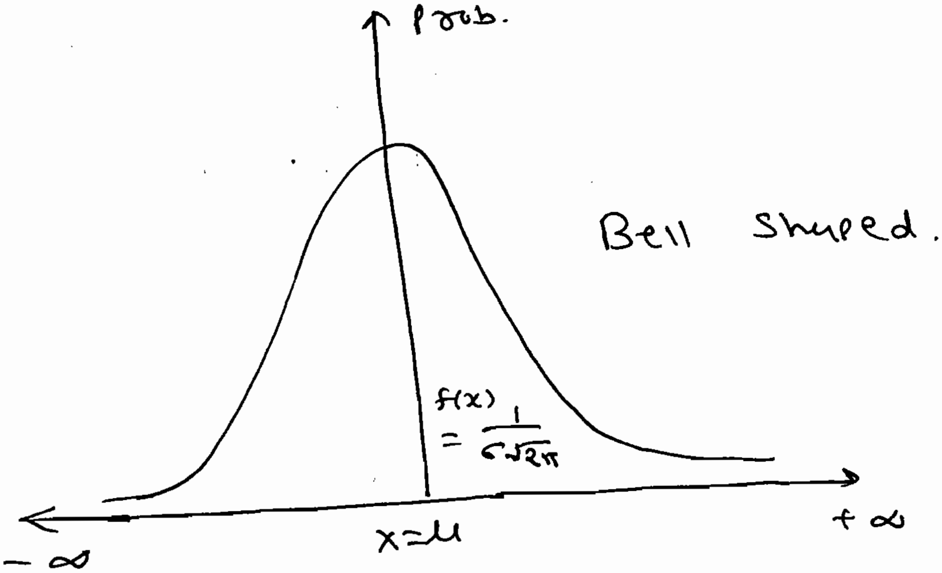
$$N(0,1) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}x^2} .$$

⇒ Mathmatecally a Standard normal R.V. is denoted with z and define as,

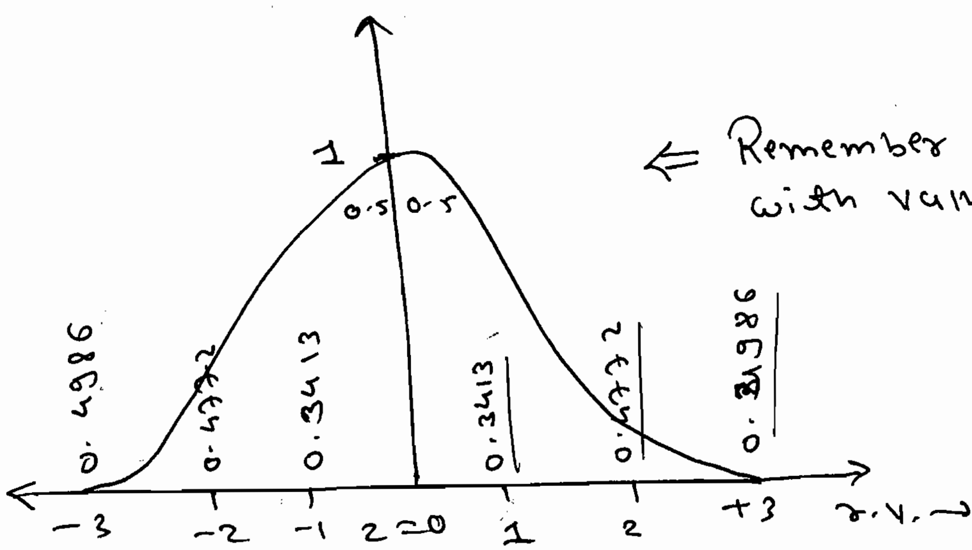
$$z = \frac{x - E(x)}{\sqrt{V(x)}} = \frac{x - \mu}{\sigma} .$$

$$-3 < z < +3$$

(1) Normal Curve:



(2) Standard normal curve.



⇐ Remember for Example with values.

* Area's Under the Normal curve.

→ $P(z \le z_0) = 0.5 + A$ $(z_0 \text{ +ve})$.

→ $P(z \le z_0) = 0.5 - A$ $(z_0 \text{ -ve})$.

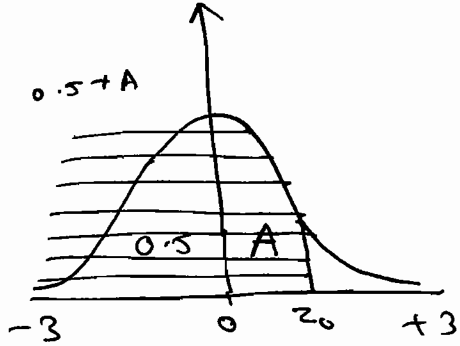
→ $P(z_1 \le z \le z_2) = A_1 + A_2$ $(z_1 \text{ -ve } \& \ z_2 \text{ +ve})$.

→ $P(z_1 \le z \le z_2) = A_2 - A_1$ $(z_1 \ \& \ z_2 \ (\text{+ve} \mid \text{-ve}))$.

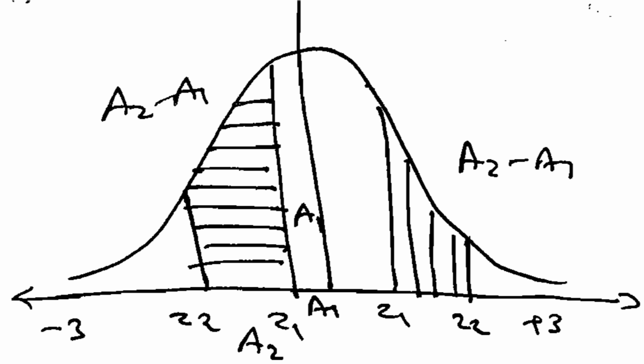
→ $P(z \ge z_0) = 0.5 + A$ $(z_0 \text{ -ve})$.

→ $P(z \ge z_0) = 0.5 - A$ $(z_0 \text{ +ve})$.

①

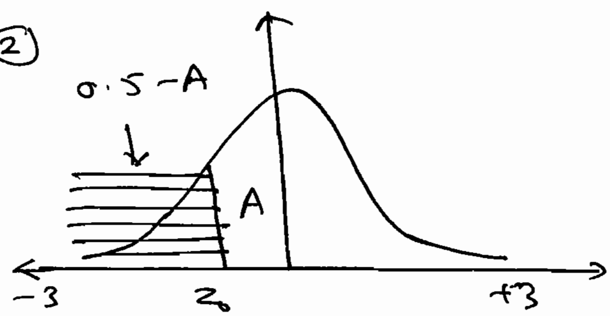


A is the Area from $z=0$ to $z=z_0$ always.

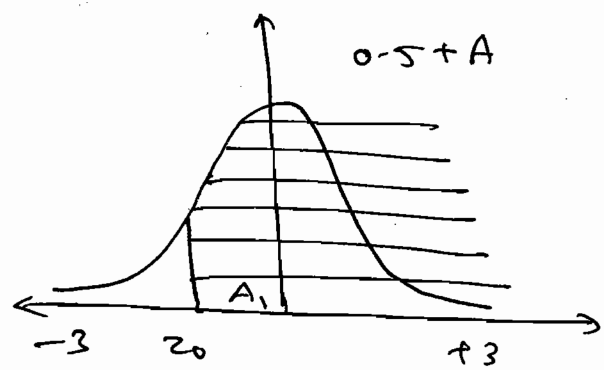


z_1 & z_2 = -ve

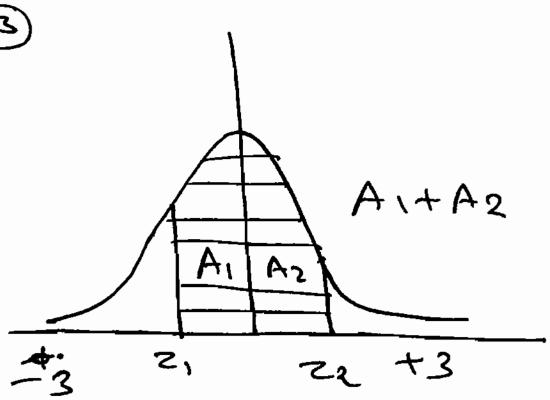
②



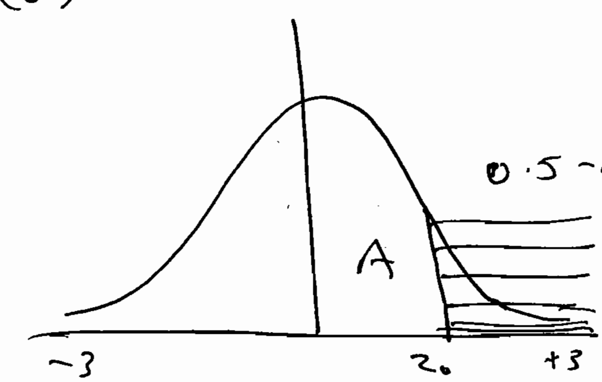
(5)



③



(6)



Ex-1 If x is normally distributed with mean $=20$ and S.D $=3.33$ find the prob. bet^n 21.11 & 26.66. The Area under curve $z=0$ to $z=0.33$ is 0.1293.

Ans: $E(x) = \mu = 20.$

$\sigma = 3.33.$

$x_1 = 21.11$

$x_2 = 26.66.$

$A \simeq 0.$

$\therefore \quad z_1 = \dfrac{x_1 - \mu}{\sigma}$

$$z_1 = \frac{21.11 - 20}{3.33} = \frac{1.11}{3.33} = 1/3.$$

$$z_2 = \frac{26.66 - 20}{3.33} = \frac{6.66}{3.33} = 2.$$

Now $P(z_1 \le z_0 \le z_2)$

$= P(0.33 \le z_0 \le 2)$

$= \cancel{A_2 + A_1}$ $A_2 - A_1$

$= \cancel{0.33 + 0.4772}$ $0.4772 - 0.1293$

$= \cancel{0.47}$

$= 0.3479.$

Ex-3 If x is distributed with $E(x) = 30$: $\sigma = 5$.
find $P(|x-30| < 5)$.

Ans: $\cancel{Z_6 = 1.5}$. $\cancel{Z_{1}}$

$E(x) = 30$, $\sigma = 5$.

$\therefore P(|x-30| < 5)$

$= P(-5 < x-30 < +5).$

$= P(\underset{x_1}{25} < x < \underset{x_2}{35})$

$$\therefore z_1 = \frac{x_1 - \mu}{\sigma} = \frac{25-30}{5} = -1.$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{35-30}{5} = +1.$$

$\therefore P(-1 < z < +1) = 0.3413 + 0.3413. = A_1 + A_1$

$\therefore P(25 < x < 35) = 0.6826.$

Ex-3 A dice is rolled 180 time use the normal dis^n find the prob. the Face 4 will turn up atleast 35 times.

Ans: Here, R.V. are independent so we can use Binomial R-V.

$\therefore$ $n = 180$ $p = 1/6$, $q = 5/6$.

$\therefore$ $E(X) = np = 180 \times 1/6 = 30$.

$V(X) = npq = 180 \times \frac{1}{6} \times \frac{5}{6} = 25$.

$\therefore$ $P(X \geq 35)$

$$z = \frac{x - \mu}{\sigma} = \frac{x - 30}{\sqrt{25}} = \frac{35 - 30}{5} = 1.$$

$\therefore$ $P(z \geq 1) = 0.5 - A$

$= 0.5 - 0.3413$

$= 0.1587.$

NOTES:

→ Sum and differences bet^n the independent normal random variables R.V. is also a normal random variables.

→ Binomial dis^n is approximation of normal if $n \to \infty$, neither the prob. is small nor the failur are large.

✩ Corelation and Regression.

⇒ Defn:

→ The relation betn the 2-D R.V. in bivariant data is known as corelation.

i.e. the changes in the one variable is affecting the changes of the other variable $\parallel^{ee}$ then those variable are known as co-variable.

* Types of the Corelation:

① **Positive corelation:**

→ If the changes in the both variable are in the same direction (increasing or decresing) then those variables are known as positively corelated.

② **Negatively corelation:**

→ If the changes in the one variable is affecting the changes in the other variable in Reverse direction then those variable are known as negatively corelated variables.

* Karl Pearson's Corelation Coeficient

Correlation Coefficient

$$\boxed{r(x,y) = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}}.$$

where $\left| \operatorname{cov}(x,y) = \frac{1}{n}\sum x \cdot y - \bar{x}\bar{y} \right|$, $-1 \le r \le +1$.

Note:

→ If x & y are independent R.V. then

$\boxed{\mathrm{cov}(x, y) = 0}$. ⇒ $\boxed{r(x, y) = 0}$.

i.e. they are highly uncorelated.

→ Corelation Co-efficient is geometrically measured with Scatter diagram.

→ It is independent of change of origin as well as independent of change of scale.

* Regression: (simple - linear).

⇒ **Defⁿ:**

→ The Linear Relationship betⁿ corelated Variables is known as regression.

⇒ **Lines of Regression:**

Y on X:

$$\boxed{y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x} \cdot (x - \bar{x})}$$

reg. coefficient (y on x)

$$\boxed{b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}}$$

X on Y:-

$$\boxed{x - \bar{x} = r \cdot \frac{\sigma_x}{\sigma_y} \cdot (y - \bar{y})}$$

reg. coefficient (x on y)

$$\boxed{b_{xy} = r \cdot \frac{\sigma_y}{\sigma_x}}$$

⇒ Properties:

→ $b_{yx} \times b_{xy} = r \cdot \frac{\sigma_y}{\sigma_x} \times r \times \frac{\sigma_x}{\sigma_y} = r^2.$

$$\therefore \boxed{r = \pm\sqrt{b_{yx} \times b_{xy}}}$$

→ $\boxed{b_{yx} > 1 \ : \ b_{xy} < 1.}$ (vice versa).

$b_{yx} = b_{xy} \Rightarrow r \cdot \frac{\sigma_y}{\sigma_x} = r \cdot \frac{\sigma_x}{\sigma_y}.$

$$\Rightarrow \boxed{\sigma_x^2 = \sigma_y^2}$$

⇒ Angle:-

$$\boxed{\theta = \tan^{-1}\left(\frac{1-r^2}{|r|} \cdot \frac{\sigma_x \cdot \sigma_y}{\sigma_x^2 + \sigma_y^2}\right).}$$

$r = 0 \Rightarrow \theta = \pi/2.$

$r = 1 \Rightarrow \theta = 0$ or π.

NOTE:

→ The Regression eqn are passes ~~process~~ to the point $\bar{x}, \bar{y}$.

→ Both the regression co-efficient must have a same sign. i.e. if both are +ve ⇒ r is +ve.
if both are −ve ⇒ r is −ve.

→ Regression co-efficient is independent of change of origin ~~horizon~~ and dependent of change of scale.

Ex-1 Regression eq. are ...

$2x + y = 1$.

1. find the value of r
2. find the means of x & y
3. if $\sigma_x = 1$ find $\sigma_y = ?$

Ans: $x + 2y = 0$. since, co-efficient of y is more than x, so y on x.

$2x + y = 1$. x on y similarly.

y on x	x on y
$\therefore x + 2y = 0$	$2x + y = 1$.
$\therefore y = -x/2$.	$\therefore x = 1/2 - y/2$
$\therefore b_{yx} = -1/2$	$\therefore b_{xy} = -1/2$.

(2) $r = \sqrt{b_{xy} \times b_{yx}}$

$= -\sqrt{\frac{1}{2} \times \frac{1}{2}}$

$$\boxed{r = -1/2}$$

Now,
$$\begin{array}{r} \cancel{2\bar{x}} + 4\bar{y} = 0 \\ \cancel{2\bar{x}} + \bar{y} = 1 \\ \hline 3\bar{y} = -1. \end{array}$$

$\boxed{\bar{y} = -1/3}$ $\boxed{\bar{x} = -1/3}$.

$(\bar{x}, \bar{y}) = (2/3, -1/3)$.

(iii) $b_{yx} = -1/2$. $\therefore -\frac{1}{2} \cdot \frac{\sigma_y}{\sigma_x} = -1/2$.

$r \cdot \frac{\sigma_y}{\sigma_x} = -1/2$. $\sigma_y = 1$.

OM

Sujal N. Patel (+~~[illegible]~~).

ECE

ACE Academy

Maths (Numerical methods).

PM 1(B)

☆ Numerical Method

* Types of Errors:

(1) Inherent error:

→ It is already existing in the data before finding the solⁿ of the problem.

→ This error occurs due to computer precision. e.g.: $A_c = \pi r^2$, $\pi = \frac{22}{7} = 3.14.$

error is exist in π.

(2) Round-off error:

→ This error occurs due to the converting the significant digit in a nearer integer.

* **Rules:**

⊙ If n^{th} place is more than half of $n^{th}+1$ place then increase the unity of the n^{th} place.

$\therefore \boxed{n^{th} > \frac{1}{2}(n+1)^{th} \uparrow}$

e.g. → $3.4678 \cong 3.468.$

⊙ If n^{th} place and $(n+1)^{th}$ place both are the odd nos. (same digit) then also increase the unity of the n^{th} place.

$\boxed{n^{th} = (n+1)^{th} = \text{odd} \uparrow}$

$3.4611 \cong 3.462.$

are the even no. (same digit) then leave the n^{th} place as it is.

e.g. $3.4622 \simeq 3.462$

(3) Truncation Error:

→ This error occurs due to discarding the terms from infinity series or power series.

→ Truncation errors are two types:

(1) Local truncation error.

(2) Propogation of Truncation error.

→ Truncation errors are more serious then Round off errors.

→ Truncation errors are associated with sol^n of the numerical $diff^n$ eq^n.

(4) Obsolute Error:

→ The difference bet^n the True value and approximate value is know as absolute Error.

$$\boxed{A \cdot E. = |x' - x|}$$

x' → approx value

x → True value

$\therefore \boxed{R.E. = \frac{|x'-x|}{x} = \frac{\quad}{T.V}}$

$\therefore \boxed{P.E. = \left|\frac{x'-x}{x}\right| \times 100 \%}$

$= \underline{RE \times 100.}$

→ This three errors are more serious in the soln Transidental eqn.

☆ Solution for the Transcedental Eqn. :-

⇒ Defn of Transcedental eqn:

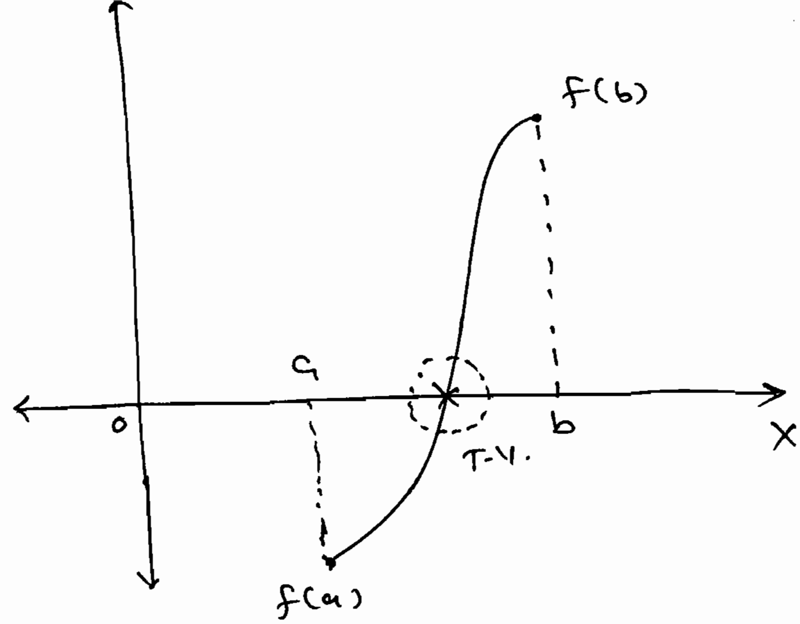
→ An eqn which involves exponetial, Trignomatric and logarithmic terms then the eqn is know as transcedental eqn.

e.g. $f(x) = x e^{x} - \cos x.$

$f(x) = \cos x - \log x + x + 2.$

⇒ Intermidiate Value Property:

→ If $f(x)$ is contn in $[a,b]$, $f(a)$, $f(b)$ are different signs ($f(a) \cdot f(b) < 0$) then there exist at least one root in the closed interval $[a,b]$.



→ In general we can find the initial approximations of the solⁿ of the Truncedental eqⁿˢ using intermidiate value property.

⇒ Rate of Convergence:

→ If $\frac{e_{i+1}}{e_i} = \frac{x_{i+1} - x}{x_i - x}$ = almost constant

then the Rate of convergence is said to be slow and order of convergences is linear or first order.

→ If $\frac{e_{i+1}}{e_i^p}$ = nearer to constant, the Rate of convergence is know as faster and order of convergences p^{th} order $(p>1)$.

(1) Bisection Method (Halving Method)

→ $\boxed{\text{Iterative Formula} = \frac{(+ve)+(-ve)}{2}}$.

* Procedure:

if $f(x)$ is cont^n in $[a,b]$ $f(a) = -ve$ and $f(b) = +ve$.

$f(a) = -ve$
$f(b) = +ve$

first approximation $= \frac{b+a}{2} = x_1$; $f(x_1) = -ve$.

∴ 2^{nd} approx $= \frac{b+x_1}{2} = x_2$: $f(x_2) = +ve$.

∴ 3^{rd} approx $= \frac{x_1+x_2}{2} = x_3$; (till accuracy)

Step-1

→ This method is guarantee to converge but very slow. since we are reaching to the true value on both sides of the polynomial.

→ Overall rate of convergence is slow Convergence and order of convergence is linear.

→ In this method we are reducing $\frac{1}{2}$ factor of error on step by step therefore the length of the interval at the n^{th} step is

$$\boxed{\frac{|b-a|}{2^n} \leq \epsilon}$$

where ϵ is small error

$$\Rightarrow \quad \frac{1-0}{2^n} \le 10^{-2}$$

$$\therefore \quad \frac{1}{2^n} \le \frac{1}{10^2}.$$

$$\therefore \quad 2^n \quad \boxed{n=7}$$

→ By using this method we can not locate complex roots of the eqn.

Ex-1 Find the third approximation the fn $f(x) = x^3 - 4x - 9$ in $[2, 3]$.

Ans:

$f(2) = 8 - 8 - 9 = -9$ (-ve).

$f(3) = 27 - 12 - 9 = +6$ (+ve).

$\therefore$ F.A. $= \frac{3+2}{2} = 2.5$: $f(2.5) = -$ve.

$\therefore$ S.A. $= \frac{3+2.5}{2} = 2.75$: $f(2.75) = +$ve.

$\therefore$ T.A. $= \frac{2.5 + 2.75}{2} = 2.625$ //

Ex-2 Find the 3rd approximation for the fn $f(x) = xe^x - 1$, $[0, 1]$.

Ans:

$f(0) = -1$, (-ve).

$\therefore f(1) = 1e^1 - 1$, (+ve)

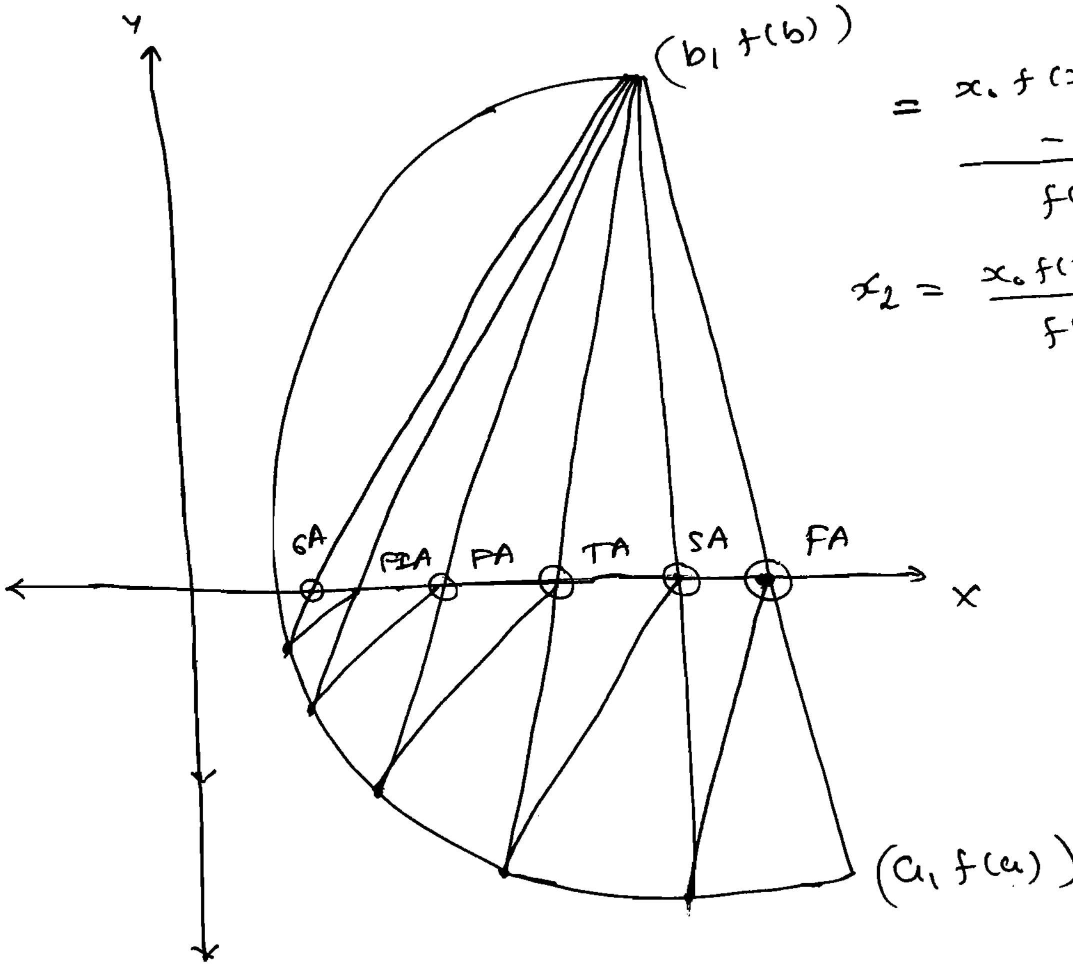
$\therefore$ F.A. $= \frac{0+1}{2} = 0.5$: $f(0.5) = -$ve.

$\therefore$ S.A. $= \frac{1+0.5}{2} = 0.75$: $f(0.75) = +$ve.

$\therefore$ T.A. $= \frac{0.5+0.75}{2} = 0.625$ //

(2) Regular false

$$\rightarrow \quad x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0)$$



$$= \frac{x_0 f(x_1) - x_0 \cancel{f(x_0)} - x_1 f(x_0) + x_0 \cancel{f(x_0)}}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}.$$

→ This method also guarantee to converge and also faster than the bisection method. since we are reaching to the T.V. on one side of the polynomial.

→ over all Rate of convergence is slow convergence and order of convergence is linear.

→ If the 1st Approximation final value is -ve till end of the problem the approximation and must be -ve. i.e. the roots is approches

→ By using this method also we can not locate complex roots of the eqn.

Ex-1 Find the 3rd Approx. for the fn
$f(x) = x\log_{10} x - 1.2.$ in $[2,3]$.

Ans:
$f(2) = 2\log_{10} 2 - 1.2 = -0.5979. \rightarrow x_0$
$f(3) = 3\log_{10} 2 - 1.2 = 0.2316. \rightarrow x_1$

$$\therefore \quad FA = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0).$$

$$\therefore \quad = 2 - \frac{3-2}{0.2316 + 0.5979} (+0.5979).$$

$$= 2.7202.$$

$$\therefore \quad f(2.7202) = -0.01729.$$

$$S.A. = 2.7202 - \frac{3 - 2.7202}{0.2316 + 0.01705} (+0.01709).$$

$$\therefore SA = 2.7401 \simeq 2.740.$$

$$\therefore \quad f(2.7401) = -0.00038.$$

$$\therefore \quad 3A = 2.7401 - \frac{3 - 2.7401}{0.2316 + 0.00038} (-0.00038)$$

$$\therefore \quad 3A = 2.7404 \simeq 2.740.$$

$f(x) = x e^x - \cos x \rightarrow [0,1]$.

$\downarrow$
Radian

Ans: $f(0) = 0 \cdot e^0 - \cos 0 = -1$. x_0

$f(1) = 2.179$. x_1

$\therefore$ $F.A = 0 - \dfrac{1-0}{2.179+1} \cdot (-1)$.

$= \dfrac{1}{3.179} = 0.3146$.

$\therefore$ $f(0.3146) = -0.5719$.

$S.A. = 0.3146 - \dfrac{1-0.3146}{2.179+0.5719} \times (-0.5719)$.

$= 0.4462$.

$\therefore$ $f(0.4462) = -0.2049$.

$3A = 0.4462 - \dfrac{1-0.4462}{2.179+0.2049} \times (-0.2049)$.

$= 0.49402$.

$\therefore$

(3) Secant Method:- (...

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n) \qquad n \geq 1$$

→ This method is no guarantee to converge since for finding the successive ~~appr~~ approximation we are using last two current itterations [(+ve, +ve), (−ve, −ve), (+ve, −ve).]

→ It for all convergence it is faster than the regular false method. (1.62 times).

→ overall rate convergence, faster convergence and order of convergeces 1.62.

→ By using this method we can locate complex roots of the eqn

Ex-1 Find the 3rd approximation for the fn $f(x) = xe^x - \cos x$ in [0,1].

Ans: $f(x) = xe^x - \cos x.$

$f(0) = -1.$

$\therefore f(1) = 2.179.$

$$\therefore FA = 1 - \frac{1-0}{2.179+1}(2.179).$$

$= 0.3146.$

$$\therefore\ SA = 0.3146 - \frac{0.3146 - 1}{-0.5719 - 2.179}\ (-0.5719).$$

$$= 0.4462\ ; \quad f(0.4462) = -0.2049.$$

$$3A = 0.4462 - \frac{0.4462 - 0.3146}{-0.2049 + 0.5719} \times (-0.2049).$$

$$\therefore\ 3A = 0.5314.$$

(4) Newton Raphson's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (\because f'(n) \neq 0).$$

→ This method is also guarantee to converge provided if the initial approximation is nearer to the true value (T.V.).
(sensitivity of the root).

→ Overall rate of convergence is faster convergence and order of convergence is quadratic (or) second order.

⇒ Meaning quadratic:
error is square of the propositional to the previous error.

→ For finding the successive approximation use the root of the functional value.

→ If the derivative of the functional value is the higher the method convergese more rapidly. otherwise very slow some times divergese.

→ If this method fails apply the Regular False method

→ This method is also known as tangent method (geomatrically).

→ This is the best method for finding the complex roots of the eq^n.

→ By using this method we can find $\sqrt{N}$, $\sqrt[3]{N}$, $\frac{1}{N}$, $\frac{1}{\sqrt{N}}$, $\frac{1}{(N)^{1/3}}$ ---- etc.

* → Square Root: $\sqrt{N}$.

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)}.$$

→ Cube Root : $\sqrt[3]{N}$.

$$\boxed{x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)}$$

→ p^{th} Root :

$$\boxed{x_{n+1} = \frac{1}{p}\left((p-1)x_n + \frac{N}{x_n^{p-1}}\right)}$$

$$\therefore \boxed{x_{n+1} = x_n(2 - N x_n).}$$

→ I. S. Root: $1/\sqrt{N}$.

$$\therefore \boxed{x_{n+1} = \frac{x_n}{2}(3 - N x_n^2).}$$

→ I. C. Root: $1/\sqrt[3]{N}$.

$$\therefore \boxed{x_{n+1} = \frac{x_n}{3}(4 - N x_n^3).}$$

→ I. p^{th} root: $\frac{1}{\sqrt[p]{N}}$

$$\therefore \boxed{x_{n+1} = \frac{x_n}{p}((p+1) - N x_n^{p}).}$$

NOTE:

→ For all the above Iterative Schema the rate of convergence is faster and order of convergence is second order.

→ The title of the formula itself is a converging point.

Ans: $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right].$

$\sqrt{12}$

$\sqrt{9}$ $\quad$ $\sqrt{16}$

$=3$ $\quad$ $=4$

$\therefore \quad x_0 = \frac{3+4}{2} = 3.5.$

$\therefore \quad F.A. = \frac{1}{2}\left[3.5 + \frac{12}{3.5}\right] = 3.4642.$

$S.A. = \frac{1}{2}\left[3.4642 + \frac{12}{3.4642}\right] = 3.4641.$

$3.A. = \frac{1}{2}\left[3.4641 + \frac{12}{3.4641}\right] = 3.4641.$

Ex-2 Find the $(10)^{1/3}$ by newton's method.

Ans:- $x_{n+1} = \frac{1}{p}\left[(p-1)\,x_n + \frac{N}{x_n^{p-1}}\right].$

$(10)^{1/3}$

$(8)^{1/3}$ $\quad$ $(27)^{1/3}$.

$\therefore \quad x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right].$

$\therefore \quad F.A. = \frac{1}{3}\left[2(2.5) + \frac{10}{(2.5)^2}\right] = 2.2.$

$S.A. = \frac{1}{3}\left[2(2.2) + \frac{10}{(2.2)^2}\right] = 2.1553.$

$3A = \frac{1}{3}\left[2(2.1553) + \frac{10}{(2.1553)^2}\right] = 2.1543.$

$x_0 = 1.$

Ans: $f(x) = x e^x - 1. \rightarrow f(1) = 1 \cdot e^1 - 1 = 1.718.$

$f'(x) = x e^x + e^x. = e^x (x+1).$

$f'(1) = 5.436.$

$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$

$\therefore \quad FA = 1 - \frac{1.718}{5.436} = 0.6839.$

$\therefore \quad \cancel{SA}. \quad f(0.6839) = 0.3552.$

$f'(0.6839) = 3.3362.$

$\therefore \quad S.A. = 0.6839 - \frac{0.3552}{3.3362}.$

$S.A. = 0.5771.$

$\therefore$

☆ Solution for Numerical (...)

⇒ Application of the diffⁿ eqⁿ:

* → Initial Value Problem: (IVP).

→ Let, the n^{th} order diffⁿ eqⁿ is

$$f(x, y, y', y'', \dots y^{n}) = 0. \quad \text{— ①. and}$$

its general solⁿ is

$$\phi(x, y, c_1, c_2, \dots c_n) = 0. \quad \text{Where,}$$

$c_1, c_2, \dots c_n$ are arbitrary constants.

→ To find the particular solⁿ for eqⁿ-① we require n condition. If this all n condition are prescribed at one point say $x = x_0$ then the diffⁿ eqⁿ-① and conditions together known as initivalue problem. It is solvable by ordinary diffⁿ eqⁿ.

* → Boundary Value Problem (BVP).

→ If the conditions are prescribe at more than at one point say $x = x_1, x_2$ then diffⁿ eq-① and condition together known as Boundary Value problem. Solvable with finite element methods.

→ If y value can be incremeant only one step at a time then the methods are known as single step methods: that means the succeeding values are incrementing with the help imidiate preeeeding value.
therefore the general rule is

New value = Old value + (slop × step size).

⇒ Multi steps Methods:

→ If the value of y is incremanting by more than one step at a time then the methods are known as Multi steps methods. This methods are also known as predictor's corrector's methods.

→ Standard form is

$$\frac{dy}{dx} = f(x,y) \; : \; y(x_0) = y_0.$$

- Single step
  - Power series
    - → Picard's
    - → Taylor's
  - Step by step
    - → Euler's
    - → R.K. methods
- Multi step.
  - → Milne's
  - → Adams

  } predictor's corrector's.

Picard's Method:

$\rightarrow \quad \frac{dy}{dx} = f(x, y) \; : \; y(x_0) = y_0.$

$\therefore \quad dy = f(x, y) \cdot dx.$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y) \cdot dx$$

$$\therefore \quad y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

Stop

$$\therefore \quad y = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

N.V.   O.V.   step size.

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

$$\therefore \quad y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$\boxed{y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.}$$

Ex-1 Solve the diff^n eq^n:

$\frac{dy}{dx} = x + y$ such that $y(0) = 1$. upto 3^{rd} A. and find $y_3(1)$.

Ans: $f(x, y) = x + y.$

$\therefore \quad y_0 = 1, \quad x_0 = 0.$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

$$\therefore y_1 = 1 + \int_0^x (x+1)\,dx.$$

$$y_1 = 1 + x + \frac{x^2}{2}.$$

$$\therefore y_2 = 1 + \int_0^x (x + y_1)\,dx$$

$$= 1 + \int_0^x 1 + 2x + \frac{x^2}{2}.\,dx.$$

$$y_2 = 1 + x + x^2 + x^3/6.$$

$$\therefore y_3 = 1 + \int_0^x (x + y_2)\,dx$$

$$= 1 + \int_0^x 1 + 2x + x^2 + x^3/6\,.dx.$$

$$y_3 = 1 + x + x^2 + x^3/3 + x^4/24.$$

Now, $y_3(1) \rightarrow$ put $x = 1$.

$$\therefore y_3 = 1 + 1 + 1 + 1/3 + 1/24.$$

$$y_3 = 81/24. = 27/8. = 3.375.$$

$\rightarrow \quad \frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$

$$\therefore \boxed{y_n = y_0 + \frac{(x-x_0)^1}{1!}(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \cdots + \frac{(x-x_0)^n}{n!}(y^n)_0}$$

where, $(y')_0 = \left(\frac{dy}{dx}\right)_{x_0, y_0}$ at initial value.

$(y'')_0 = \left(\frac{d^2y}{dx^2}\right)_{x_0, y_0}.$

$\vdots$

$(y^n)_0 = \left(\frac{d^ny}{dx^n}\right)_{x_0, y_0}.$

NOTE:

$\rightarrow$ In this method the successive approximations ~~are~~ are representing with successive order of derivatives.

for e.g. 1^{st} A $= y'$

2^{nd} A $= y''$ and so on.

Ex-1 Solve the diff^n eq^n.

$$\frac{dy}{dx} = y^2 + 2xe^x + e^{2x}. \quad \text{S.T. } y(0)=1, \text{ find } y_3 = ? \text{ and } y(2).$$

Ans:. here, $y' = y^2 + 2xe^x + e^{2x}.$ $\quad x_0 = 0, y_0 = 1$

$\therefore (y')_{(0,1)} = y^2 + 2xe^x + e^{2x} = 1 + 2\cdot 0\cdot e^0 + e^0 = 2.$

$(y'')_{(0,1)} = 2yy' + 2xe^x + 2e^x + 2e^{2x}$

$= 1\cdot 2 + 2\cdot 0\, e^0 + 2\cdot e^0 + 2\cdot e^0$

$$(y''')_{(0,1)} = 2y\cdot y'' + 4y' \cdots + 4e^x$$

$$= 2.1.8 + 2.4.2 + 2.0e^0 + 2e^0 + 2e^0 + 4e^0$$

$$= 32$$

$$\therefore \quad y_1 = 1 + \frac{(x-0)^1}{1!}(2) + \frac{(x-0)^2}{2!}(8) + \frac{(x-0)^3}{3!}(32).$$

$$y = 1 + 2x + 4x^2 + 16/3\, x^3$$

$$\therefore \quad y(2) = 1 + 4 + 16 + 128/3$$

$$\therefore \quad y(2) = 191/3.$$

$$\rightarrow \frac{dy}{dx} = f(x,y) \quad : \quad y(x_0) = y_0.$$

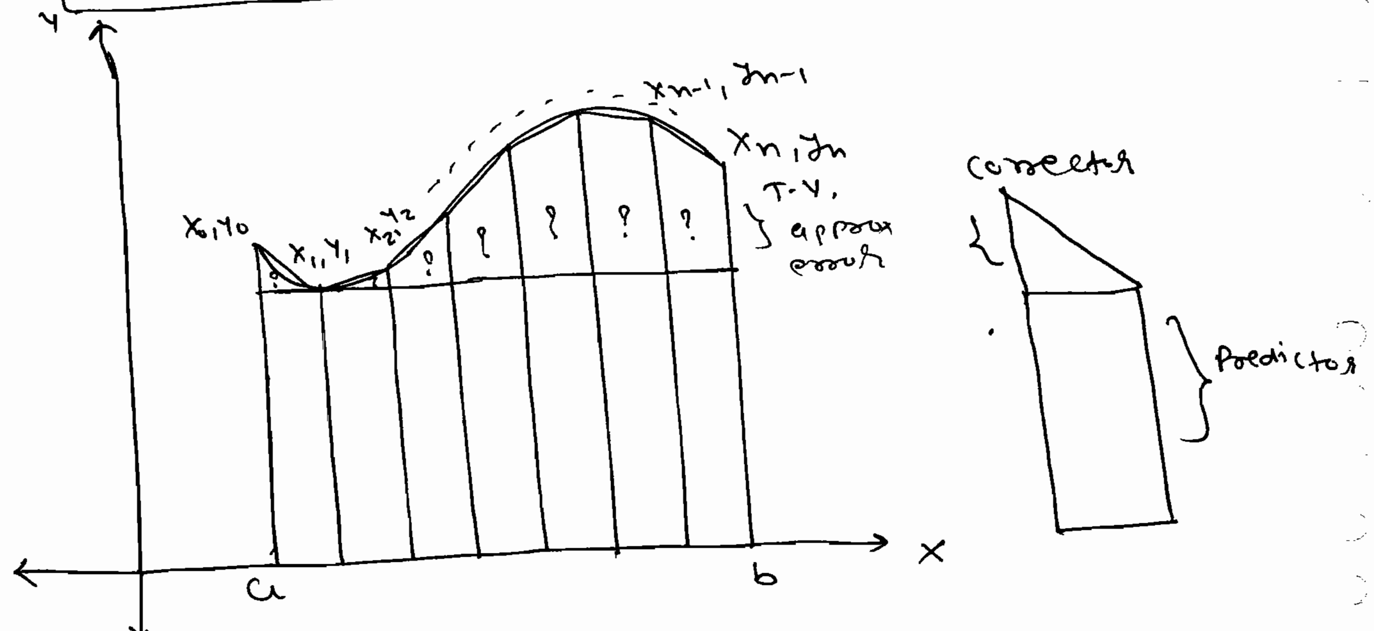
$$\therefore \boxed{y_1 = y_0 + h\,f(x_0, y_0).}$$

$$y_2 = y_1 + h\,f(x_1, y_1).$$

$$\therefore y_3 = y_2 + h\,f(x_2, y_2).$$

$$\vdots$$

$$\boxed{y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1}).}$$



→ Geometrically in the simple Euler's method we are ~~going~~ joing the points under the curve by taking a straight line. Sometimes the sequence of straight lines are deviating from the actual sol^n.

To overcome this in modified Euler's method we are joing the point under the curve by taking a curvature.

→ Simple Euler's method is application
for the Rectangular rule.

→ Modified Euler's method is application
for the Trapezoidal Rule.

→ The order of truncation Error in the
Euler's method is $O(h^2) \Rightarrow$ order of h^2.

→ The degree of the polynomial in the
Euler's method is 1^{st} degree (straight line).

→ This method is also known as predector's
and corrector's (single step) method.
Continue the corrector's itteration until
the accuracy.

* Modified Euler's Formula or Improved Euler's Formula.

→ $\frac{dy}{dx} = f(x, y)$: $y(x_0) = y_0$.

$$\boxed{\begin{aligned}
y_{1,p} &= y_0 + h f(x_0, y_0). \\
y_{1,c} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_{1,p})\right]. \\
y_{1,(c)^1} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_{1,c})\right]. \\
y_{1,(c)^2} &= y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_{1,(c)^1})\right]
\end{aligned}}$$

⋮

till accuracy.

Ex-1 Apply the [illegible]

the $\frac{dy}{dx} = x + y$, $y(0) = 1$. find $y(0.1)$ in steps of 0.02.

Ans: $f(x, y) = x + y$,

$x_0 = 0$,

$y_0 = 1$,

$h = 0.02$.

x	y	$f(x, y) = x + y$	N.V.
$x_0 = 0$	$y_0 = 1$	$f(x_0, y_0) = 0 + 1 = 1$	$y_1 = 1 + 0.02(1) = 1.02$
$x_1 = 0.02$	$y_1 = 1.02$	$f(x_1, y_1) = 1.02 + 0.02 = 1.04$	$y_2 = 1.02 + 0.02(1.04) = 1.04$
$x_2 = 0.04$	$y_2 = 1.04$	$f(x_2, y_2) = 1.04 + 0.04 = 1.08$	$y_3 = 1.04 + 0.02(1.08) = 1.06$.
$x_3 = 0.06$	$y_3 = 1.06$	$f(x_3, y_3) = 1.06 + 0.06 = 1.12$	$y_4 = 1.06 + 0.02(1.12) = 1.08$.
$x_4 = 0.08$	$y_4 = 1.08$	$f(x_4, y_4) = 1.08 + 0.08 = 1.16$	$y_5 = 1.08 + 0.02(1.16)$ $y_5 = 1.1$
$\boxed{x_5 = 0.1}$	$\boxed{y_5 = 1.1}$		

$\therefore$ $y(0.1) = 1.1$.

Ex 2 Apply the modified Euler's method

$\frac{dy}{dx} = x + y$, s.t. $y(0) = 1$, find $y(1)$.

Ans. $f(x, y) = x + y$.

$x_0 = 0$

$y_1 = 1$

$\therefore$ $x_0 + h = 1$

$0 + h = 1$

$\boxed{h = 1}$

x	y	$f(x,y) = x+y$	$\frac{f(x_0,y_0) + f(x_1,y_1)}{2}$	y
$x_0 = 0$	$y_0 = 1$	$f(x_0, y_0) = 1$	—	$y_{1,p} = 1 + 1(1) = 2$
$x_0 = 1$	$y_{1,p} = 2$	$f(x_1, y_1) = 3$	$\frac{1}{2}(1+3) = 2$	$y_{1,c}^{1} = 1 + 1(2) = 3$
$x_0 = 1$	$y_{1,c}^{2} = 3$	$f(x_1, y_1) = 4$	$\frac{1}{2}(1+4) = 2.5$	$y_{1,c}^{2} = 1 + 1(2.5) = 3.5$
$x_0 = 1$	$y_{1,c}^{3} = 3.5$	$f(x_1, y_1) = 4.5$	$\frac{1}{2}(1+4.5) = 2.75$	$y_{1,c}^{3} = 1 + 1(2.75) = 3.75$
$x_0 = 1$	$y_{1,c}^{4} = 3.75$	$f(x_1, y_1) = 4.75$	$\frac{1}{2}(1+4.75) = 2.8$	$y_{1,c}^{4} = 1 + 1(2.8) = 3.8$

$\therefore\ y(1) = 3.8$

$\rightarrow \quad \frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0.$

$$\therefore \quad y_1 = y_0 + k.$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4].$$

$$\therefore \quad k_1 = h[f(x_0, y_0)]$$

$$k_2 = h[f(x_0 + h/2, y_0 + k_1/2]$$

$$k_3 = h[f(x_0 + h/2, y_0 + k_2/2)]$$

$$k_4 = h[f(x_0 + h, y_0 + k_3)].$$

Ex-1 Solve the $\frac{dy}{dx} = x + y$ s.t $y(0) = 1$, find $y(0.2)$.

Ans: $f(x, y) = x + y$

$x_0 = 0$

$y_0 = 1$

$x_0 + h = 0.2$

$h = 0.2$

$\therefore$ $\cancel{k_1 = hf(x_0 + h, y_0 + h)}$

$\cancel{k_2 = 0.2 f(0.2, 1.2) = 0.24}.$

$k_1 = hf(x_0, y_0) = 0.2(0+1) = 0.2$

$k_2 = 0.2 f(0.1, 1.1) = 0.2(1.2) = 0.24.$

$k_3 = 0.2 f(0.1, 1.12) = 0.2(1.22) = 0.244.$

$\therefore k_4 = 0.2 f(0.2, 1.244) = 0.2(1.444) = 0.2888.$

$\therefore \frac{dy}{dx} = f(x, y) \quad : \quad y(x_0) = y_0.$

$\therefore$

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right].$$

$$y_{4,c} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_{4,p}\right].$$

$$y_{4,c}^{(1)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_{4,c}\right].$$

till accuracy.

→ Origin of this method is Newton's Forward Interpolation's Formula (Simpson's rule):

→ In this method the 1st iteration is 4th approximation. For evaluting this we need starting three iteration's (y_1, y_2, y_3) the which can be generate any one of single step methods.

→ Corrector's iteration has to be continue until the accuracy.

→ R.K. method's is made for smaller values of h and Milne's method for the larger values of h.

$$\rightarrow \frac{dy}{dx} = f(x,y), \quad : y(x_0) = y_0.$$

$$y_{1,p} = y_0 + \frac{h}{24}\left[55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right]$$

$$y_{1,c} = y_0 + \frac{h}{24}\left[9 f_{1,p} + 19 f_0 - 5 f_{-1} + f_{-2}\right].$$

$$y_{1,c}^{(1)} = y_0 + \frac{h}{24}\left[9 f_{1,c} + 19 f_0 - 5 f_{-1} + f_{-2}\right]$$

till accuracy

Initial

	x_{-3}	y_{-3}	f_{-3}
1st	x_{-2} $x_{-3}+h$	y_{-2}	f_{-2}
2nd	x_{-1} $x_{-3}+2h$	y_{-1}	f_{-1}.
3rd.	x_0 $x_{-3}+3h$	y_0	f_0

→ Origin of this method is Newton's Backward interpolation formula.

→ In this method also the first approximation is 4th iteration (y_1). For evaluating this we needs starting three iteration (y_{-2}, y_{-1}, y_0) which can be generate using any one of the single step methods.

→ Continue the corrector's itterations until the accuracy.

→ This is the best method for finding the stable sol^n of non-linear diff^n eq^{n}s.